CHOOSING A SHELTERED MIDDLE PATH

PIOTR MINC

Abstract. We say that point \( x \in \mathbb{R}^2 \) is sheltered by a continuum \( S \subset \mathbb{R}^2 \) if \( x \) does not belong to the unbounded component of \( \mathbb{R}^2 \setminus S \). Suppose that points \( a \) and \( b \) are the endpoints of each of three arcs \( A_0, A_1 \) and \( A_2 \) contained in \( \mathbb{R}^2 \). We prove that there is an arc \( B \subset A_0 \cup A_1 \cup A_2 \) with its endpoints \( a \) and \( b \) such that each point of \( B \) is sheltered by the union of each two of the arcs \( A_0, A_1 \) and \( A_2 \).

1. Introduction

A continuum is a compact and connected metric space. An arc is a continuum homeomorphic to the closed interval \([0, 1]\) contained in the real line \( \mathbb{R} \). We say that a point \( x \) of the plane \( \mathbb{R}^2 \) is sheltered by a continuum \( S \subset \mathbb{R}^2 \) if \( x \) does not belong to the unbounded component of \( \mathbb{R}^2 \setminus S \).

We will make the following very intuitive observation. Suppose that three arcs (paths) in the plane are leading from a point \( a \) to a point \( b \). Then, in the maze of the three paths, we may choose a middle way from \( a \) to \( b \) so that each point of the new path is sheltered by any two of the original three paths. More precisely, we will prove the following theorem.

Theorem 1.1. Suppose that points \( a \) and \( b \) are the endpoints of each of three arcs \( A_0, A_1 \) and \( A_2 \) contained in \( \mathbb{R}^2 \). Then there is an arc \( B \subset A_0 \cup A_1 \cup A_2 \) with its endpoints \( a \) and \( b \) such that each point of \( B \) is sheltered by the union of each two of the arcs \( A_0, A_1 \) and \( A_2 \).

Our proof of the theorem is elementary. In the next section, we prove Proposition 2.6, a slightly stronger version of the theorem for piece-wise linear arcs that are in general position (defined in 2.1). If the arcs \( A_0, A_1 \) and \( A_2 \) are piece-wise linear, the set \( \mathbb{R}^2 \setminus (A_i \cup A_j) \) has finitely many components for \( i, j = 0, 1, 2 \). For each component \( V \) of \( \mathbb{R}^2 \setminus (A_i \cup A_j) \), we calculate the number times the arcs \( A_i \) and \( A_j \) need to be crossed to reach \( V \) from the unbounded component. For a given \( V \), this number of crossing is either always odd or always even (Proposition 2.3). This standard calculus gives us a tool to prove the theorem for piece-wise linear arcs in general position. We then complete (in Section 3) the proof of 1.1 by approximating arbitrary \( A_0, A_1 \) and \( A_2 \) with piece-wise linear arcs.

A topological space \( X \) is arcwise connected if for any two distinct points \( a, b \in X \) there is an arc \( A \subset X \) containing both \( a \) and \( b \). (Note that we consider the empty space and a space consisting of a single point to be arcwise connected.) \( X \) is simply connected if it is arcwise connected and has trivial fundamental group, \( \pi_1(X) = 1 \).

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The following corollary is a simple consequence of Theorem 1.1 (see Section 3 for the proof).

**Corollary 1.2.** Let \(X_0, X_1\) and \(X_2\) be simply connected plane continua such that \(X_i \cap X_j\) is arcwise connected for each \(i, j \in \{0, 1, 2\}\). Then, \(X_0 \cap X_1 \cap X_2\) is arcwise connected.

The statement of the corollary is routinely used as the initial step in inductive proofs of the Topological Helly Theorem (see [2] for more information). However, S. A. Bogatyi observed in an recent article [1, p. 399] that no complete proof could be found in the literature. Subsequently, U. H. Karimov and D. Repovš [2, Theorem 1.2] gave a weaker version of the statement proving that \(X_0 \cap X_1 \cap X_2\) is cell-like connected (instead arcwise connected). Very recently, after this paper had been submitted, E. D. Tymchatyn and V. Valov [6, Proposition 1.2] gave another, independent proof of the statement equivalent to Corollary 1.2.

2. **Piece-wise linear arcs in the plane**

Suppose \(L'\) and \(L''\) are two arcs in the plane. Suppose also that a point \(p\) belongs to the intersection of the interiors of \(L'\) and \(L''\). We say that \(L'\) and \(L''\) cross each other at \(p\) if there is a topological closed disk \(D \subset \mathbb{R}^2\) such that

- \(p\) is in the interior of \(D\),
- \(L' \cap L'' \cap D = \{p\}\),
- the endpoints of \(L'\) and \(L''\) are not in \(D\),
- the boundary of \(D\) intersects each of the arcs \(L'\) and \(L''\) at exactly two points, and
- \(\text{bd} (D) \cap L'\) separates \(\text{bd} (D)\) between the two points of \(\text{bd} (D) \cap L''\).

**Definition 2.1.** Suppose that \(L_0, L_1, \ldots, L_n\) is a finite collection of piece-wise linear arcs in the plane. We say that the arcs \(L_0, L_1, \ldots, L_n\) are in general position provided that the following two conditions are satisfied.

1. If \(p \in L_i \cap L_j \cap L_k\) for some three distinct \(i, j, k = 0, \ldots, n\), then \(p\) is a common endpoint of \(L_i, L_j\) and \(L_k\).
2. If \(p \in L_i \cap L_j\) for some two distinct \(i, j = 0, \ldots, n\), and \(p\) is an interior point of \(L_i\), then \(p\) is an interior point of \(L_j\), and the arcs \(L_i\) and \(L_j\) cross each other at \(p\). (Observe that it follows that \(L_i \cap L_j\) is finite.)

**Definition 2.2.** Let \(A\) and \(B\) be two sets such that \(A \cap B\) is finite. By \(\ell (A, B)\) we denote the number of points in the intersection \(A \cap B\).

**Proposition 2.3.** Let \(L'\) and \(L''\) be two piece-wise linear arcs in general position sharing their endpoints \(a\) and \(b\). Take \(u, v \in \mathbb{R}^2 \setminus (L' \cup L'')\). Denote by \(A\) the collection containing each piece-wise linear arc \(A\) with its endpoints \(u, v\), and such that \(A, L'\) and \(L''\) are in general position. Then \(\ell (A_1, L' \cup L'') - \ell (A_2, L' \cup L'')\) is even for each \(A_1, A_2 \in A\).

**Proof.** For any two points \(s, t \in \mathbb{R}^2\), let \(\langle s, t \rangle\) denote the straight-linear segment between \(s\) and \(t\). We will state the following claim leaving its easy proof to the reader.

**Claim 2.3.1.** For any \(A \in \mathcal{A}\), let \(a_0, a_1, \ldots, a_n\) be an increasing sequence inside the arc \(A\) ordered from \(u\) to \(v\) such that the closure of each component of \(A \setminus \{a_0, a_1, \ldots, a_n\}\) is a straight-linear segment. Then, there is a positive \(\epsilon\) such that if...
Let one end of \( A' \) be \( \langle u, a'_0 \rangle \cup \bigcup_{i=1}^{n} \langle a'_{i-1}, a'_i \rangle \cup \langle a'_n, v \rangle \) with \( |a'_i - a_i| < \epsilon \) for \( i = 0, \ldots, n \), then \( A' \in A \) and \( \ell(A', L' \cup L'') = \ell(A, L' \cup L'') \). (\( \epsilon \) depends on the choice of \( a_0, a_1, \ldots, a_n \).)

Take \( A_1, A_2 \in A \). Using Claim 2.3.1 one can find an arc \( A'_1 \in A \) such that \( \ell(A'_1, L' \cup L'') = \ell(A_1, L' \cup L'') \) and the arcs \( A'_1, A_2, L' \) and \( L'' \) are in general position. We may, therefore, replace \( A_1 \) by \( A'_1 \) and assume without loss of generality that the arcs \( A_1, A_2, L' \) and \( L'' \) are in general position.

For \( i = 1, 2 \), and any two points \( s, t \in A_i \), we will denote by \([s, t]_i \) the subarc of \( A_i \) with the endpoints \( s \) and \( t \). We will assume that the arc \([s, t]_i \) is oriented from \( s \) to \( t \). We will now select a finite collection of points \( \{u_0, u_1, \ldots, u_m\} \subset A_1 \cap A_2 \).

Set \( u_0 = u \). Suppose that for some positive integer \( j \), a point \( u_{j-1} \neq v \) has been defined. Define \( u_j \) to be the first point in the oriented arc \([u_{j-1}, v]_1 \) that belongs to \([u_{j-1}, v]_2 \setminus \{u_{j-1}\} \). Since the intersection of \( A_1 \) and \( A_2 \) is finite, this construction has to end with some \( u_m = v \). Observe that \( S_j = [u_{j-1}, u_j]_1 \cup [u_{j-1}, u_j]_2 \) is a simple closed curve for each \( j = 1, \ldots, m \). Since the arcs \( A_1, A_2, L' \) and \( L'' \) are in general position, \( u_{j-1} \) and \( u_j \) do not belong to \( L' \cup L'' \). As we travel from \( a \) to \( b \) along \( L' \) and then return along \( L'' \), each time we intersect \( S_j \), we are passing from one component of the complement of \( S_j \) to another. Since we start and end at the same point, we must intersect \( S_j \) an even number of times. Thus, \( \ell([u_{j-1}, u_j]_1 \cup [u_{j-1}, u_j]_2) \) is even for each \( j = 1, \ldots, m \).

Since \( A_1 = \bigcup_{j=1}^{m} [u_{j-1}, u_j]_1 \) and \( A_2 = \bigcup_{j=1}^{m} [u_{j-1}, u_j]_2 \) for \( i = 1, 2 \), \( \ell(A_1, L' \cup L'') + \ell(A_2, L' \cup L'') \) is even. Consequently, \( \ell(A_1, L' \cup L'') - \ell(A_2, L' \cup L'') \) is even. \( \square \)

**Definition 2.4.** Let \( L' \) and \( L'' \) be two piece-wise linear arcs in general position sharing their endpoints. Let \( U \) denote the unbounded component of \( \mathbb{R}^2 \setminus (L' \cup L'') \) and let \( V \) be an arbitrary component of \( \mathbb{R}^2 \setminus (L' \cup L'') \). Let \( A \) be a piece-wise linear arc \( A \) such that

1. \( A, L' \) and \( L'' \) are in general position, and
2. one end of \( A \) is in \( U \) and the other is in \( V \).

We will say that \( V \) is an even component of the complement of the arcs \( L' \) and \( L'' \) if there is a piece-wise linear arc \( A \) such that \( \ell(A, L' \cup L'') \) is even. We will say that \( V \) is an odd component of the complement of \( L' \) and \( L'' \) if it is not even.

Observe that it follows from Proposition 2.3 that Definition 2.4 does not depend on the choice of the arc \( A \). The next proposition is also an easy consequence of Proposition 2.3.

**Proposition 2.5.** Let \( I, L' \) and \( L'' \) be three piece-wise linear arcs in general position such that

1. \( L' \) and \( L'' \) share their endpoints, and
2. the intersection \( I \cap (L' \cup L'') \) is a single point in the interior of \( I \),

then one endpoint of \( I \) lies in an even component of \( \mathbb{R}^2 \setminus (L' \cup L'') \) and the other endpoint of \( I \) is in an odd component of \( \mathbb{R}^2 \setminus (L' \cup L'') \).

**Proposition 2.6.** Suppose \( L_0, L_1 \) and \( L_2 \) are three arcs in general position sharing common endpoints \( a \) and \( b \). For each \( i = 0, 1, 2 \), let \( i' \) and \( i'' \) be such that \( \{i, i', i''\} = \{0, 1, 2\} \). Denote by \( D_i \) the union of components of \( L_i \setminus (L_{i'} \cup L_{i''}) \) that are contained in odd components of \( \mathbb{R}^2 \setminus (L_{i'} \cup L_{i''}) \). Let \( D \) be the closure of \( D_0 \cup D_1 \cup D_2 \). Then, there exists a component \( C \) of \( D \setminus \{a, b\} \) such that \( \overline{C} \) is an arc with endpoints \( a \) and \( b \).
Proof. The proposition is illustrated by Figures 1 and 2. The components of $D_0 \cup D_1 \cup D_2$ are indicated by arrows, with those contained in $C$ indicated further by arrows with triangular arrowheads and square tails.

For $i = 0, 1, 2$, let $E_i$ be an open straight-linear segment contained in $L_i \setminus (L_i' \cup L_i'')$ such that $a \in \text{cl} (E_i)$. We will first prove the following claim.

Claim 2.6.1. $D$ contains either exactly one or all three of the arcs $E_0, E_1$ and $E_2$.

There exists a triangle $T$ containing $a$ in its interior such that $T \cap (L_0 \cup L_1 \cup L_2) = T \cap (E_0 \cup E_1 \cup E_2)$. Denote by $B$ the component of $\mathbb{R}^2 \setminus (T \cup L_0 \cup L_1 \cup L_2)$ that does not contain $e_0$. Take a point $v \in H$. Let $u$ be a point in the unbounded component of $\mathbb{R}^2 \setminus (T \cup L_0 \cup L_1 \cup L_2)$. Since $L_0 \cup T$ does not separate the plane, there is an arc $K$ with endpoints $u$ and $v$ such that $K \cap L_0 = \emptyset$, $K \cap T = \{v\}$, and $K, L_1$ and $L_2$ are in general position. It follows that, for each $j = 1, 2$, the arcs $K, L_0$ and $L_j$ are in general position. Since $v$ and $E_1$ are in the same component of $\mathbb{R}^2 \setminus (L_0 \cup L_2)$, it follows that

(i) $E_1$ is contained in an even component of the complement of $L_0$ and $L_2$ if and only if $\ell (K, L_2)$ is even.

Similarly,

(ii) $E_2$ is contained in an even component of the complement of $L_0$ and $L_1$ if and only if $\ell (K, L_1)$ is even.

Observe that $v$ is in an even component of the complement of the arcs $L_1$ and $L_2$ if and only if $\ell (K, L_1) + \ell (K, L_2)$ is even. Let $I \subset B$ be the arc with endpoints $v$ and $e_0$, and containing $e_1$. Clearly, $I, L_1$ and $L_2$ are in general position and $I \cap (L_1 \cup L_2) = \{e_1\}$. It follows from Proposition 2.5 that

(iii) $E_0$ is contained in an odd component of the complement of $L_1$ and $L_2$ if and only if $\ell (K, L_1) + \ell (K, L_2)$ is even.

We can now infer 2.6.1 from (i)–(iii) by considering the following four cases:

Case (1): both $\ell (K, L_1)$ and $\ell (K, L_2)$ are even. In this case only $E_0$ (out of $E_0, E_1$ and $E_2$) is contained in $D$.

Case (2): $\ell (K, L_1)$ is odd and $\ell (K, L_2)$ is even. In this case only $E_2$ is contained in $D$. 
Case (3): $\ell(K, L_1)$ is even and $\ell(K, L_2)$ is odd. In this case only $E_1$ is contained in $D$.

Case (4): both $\ell(K, L_1)$ and $\ell(K, L_2)$ are odd. In this case all $E_0$, $E_1$ and $E_2$ are contained in $D$ (see Figure 2).

Claim 2.6.2. Each component of $D \setminus \{a, b\}$ is an 1-manifold.

Take an arbitrary point $d \in D \setminus \{a, b\}$. We will show that some neighborhood of $d$ in $D$ is an open arc. This is obvious if $d \in D_0 \cup D_1 \cup D_2$, so we may assume that $d$ belongs to the intersection to two of the arcs $L_0$, $L_1$ and $L_2$. We may assume without loss of generality that $d \in L_0 \cap L_1$. Since the arcs $L_0$, $L_1$ and $L_2$ are in general position, there exist two open arcs $I_0 \subset L_0$ and $I_1 \subset L_1$ containing $d$ in such that $\text{cl}(I_0) \cap (L_1 \cup L_2) = \{d\}$ and $\text{cl}(I_1) \cap (L_0 \cup L_2) = \{d\}$. Apply Proposition 2.5 with $I = \text{cl}(I_0)$, $L' = L_1$ and $L'' = L_2$ to get the result that exactly one component of $I_0 \setminus \{d\}$ is contained in $D$. We will denote this component by $G_0$. Similarly, let $G_1$ be the only component of $I_1 \setminus \{d\}$ is contained in $D$. Observe that $G = G_0 \cup \{d\} \cup G_1$ is an open arc such that $(I_0 \cup I_1) \cap D = G$. Hence, the claim is true.

Claim 2.6.3. If $\bar{C}$ is a component of $D \setminus \{a, b\}$, then either

1. $\text{cl} \left( \bar{C} \right)$ is an arc with the endpoints $a$ and $b$, or

2. $\bar{C}$ contains an even number (possibly 0) of the arcs $E_0$, $E_1$ and $E_2$.

Suppose $\bar{C}$ contains (at least) one of the arcs $E_0$, $E_1$ and $E_2$. Then, either

(*) \[ \text{cl} \left( \bar{C} \right) = \{a\} \cup \bar{C} \cup \{b\}, \]

or

(**) \[ \text{cl} \left( \bar{C} \right) = \{a\} \cup \bar{C}. \]

If (*), it follows from 2.6.2 that $\text{cl} \left( \bar{C} \right)$ is an arc is an arc with the endpoints $a$ and $b$. If (**), it follows from 2.6.2 that $\{a\} \cup \bar{C}$ is a simple closed curve, and, therefore, $\bar{C}$ contains (exactly) two of the arcs $E_0$, $E_1$ and $E_2$. Thus, the claim is true.

By 2.6.1, one of the arcs $E_0$, $E_1$ and $E_2$, say $E_0$, is contained in $D \setminus \{a, b\}$. Let $C_0$ be the component of $D \setminus \{a, b\}$ containing $E_0$. By 2.6.3, either the proposition is true (with $C = C_0$), or $C_0$ must contain two of the arcs $E_0$, $E_1$ and $E_2$. So, we
may assume without loss of generality that $E_0 \cup E_1 \subset C_0$ and $E_2 \cap C_0 = \emptyset$. Let $C$ be the component be the component of $D \setminus \{a, b\}$ containing $E_2$. Since $C$ contains only one of the arcs $E_0$, $E_1$ and $E_2$, the proposition follows from 2.6.3.

The following corollary is a weakened restatement of Proposition 2.6.

**Corollary 2.7.** Suppose $L_0$, $L_1$ and $L_2$ are three arcs in general position sharing common endpoints $a$ and $b$. Then there is an arc $C \subset L_0 \cup L_1 \cup L_2$ with its endpoints $a$ and $b$ such that each point of $C$ is sheltered by the union of each two of the arcs $L_0$, $L_1$ and $L_2$.

3. Proof of Theorem 1.1

**Proof of Theorem 1.1.** Suppose $W \subset \mathbb{R}^2$ and $r$ is a positive real number. Let $B(W, r)$ denote the set \( \{ x \in \mathbb{R}^2 \mid \exists w \in W \ x - w < r \} \).

For each positive integer $n$ and for each $i = 0, 1, 2$, we can find a piece-wise linear arc $L_i^{(n)} \subset B(A_i, \frac{1}{2^n})$ with its endpoints $a$ and $b$. We can then approximate the arcs $L_0$, $L_1$ and $L_2$ by three piece-wise linear arcs $L_0^{(n)}$, $L_1^{(n)}$ and $L_2^{(n)}$ such that

1. $a$ and $b$ are the endpoints of each of the arcs $L_0^{(n)}$, $L_1^{(n)}$ and $L_2^{(n)}$,
2. $L_i^{(n)} \subset B(A_i, 1/n)$ for each $i = 0, 1, 2$, and
3. the arcs $L_0^{(n)}$, $L_1^{(n)}$ and $L_2^{(n)}$ are in general position.

By 2.7, there is an arc $C^{(n)} \subset L_0^{(n)} \cup L_1^{(n)} \cup L_2^{(n)}$ with its endpoints $a$ and $b$ such that each point of $C^{(n)}$ is sheltered by the union of each two of the arcs $L_0^{(n)}$, $L_1^{(n)}$ and $L_2^{(n)}$. By replacing, if necessary, the sequence $\{C^{(n)}\}$ by its infinite subsequence, we may assume that $\{C^{(n)}\}$ converges (in the Hausdorff distance) to a continuum $C$. Observe that

$C \subset A_0 \cup A_1 \cup A_2$.

Let $S$ be an open ball containing $\bigcup_{i=0}^2 \bigcup_{n=1}^\infty \left(A_i \cup L_i^{(n)}\right)$. We will now observe that every point of $C$ is sheltered by $A_0 \cup A_1$. Suppose, to the contrary, that there were an arc $I \subset \mathbb{R}^2 \setminus (A_0 \cup A_1)$ intersecting $C$ and the complement of $S$. Then, there would exist an open and connected set $G$ containing $I$ such that $\text{cl}(G) \cap (A_0 \cup A_1) = \emptyset$. Since $L_1^{(n)} \subset B(A_1, 1/n)$, there would exist an integer $n$ such that $G \cap \left( L_0^{(n)} \cup L_1^{(n)} \right) = \emptyset$ and $G \cap C^{(n)} \neq \emptyset$. Consequently, there would be a piece-wise arc $K \subset G$ with one end in $C^{(n)}$ and the other in the complement of $S$. This would contradict the choice of $C^{(n)}$. Thus, we have proved that every point of $C$ is sheltered by $A_0 \cup A_1$. Similarly, we may prove that every point of $C$ is sheltered by the union of each two of the arcs $A_0$, $A_1$ and $A_2$.

Since $a, b \in C^{(n)}$ for each $n$, we have the result that $a, b \in C$. By [5, Theorem 2, p. 39], $C$ is locally connected. Every locally connected continuum is arcwise connected, see for example [4, 8.23, p.130]. Therefore, there exists an arc $B \subset C$ satisfying the conclusion of Theorem 1.1.

**Proof of Corollary 1.2.** If $X_0 \cap X_1 \cap X_2$ is empty or contain only one point then the corollary is vacuously true. Let $a$ and $b$ be two arbitrary distinct points of the intersection $X_0 \cap X_1 \cap X_2$. For each $i = 0, 1, 2$, let $i'$ and $i''$ be such that $(i, i', i'') = \{0, 1, 2\}$. For each $i = 0, 1, 2$, let $A_i \subset X_{i'} \cap X_{i''}$ be an arc with the endpoints $a$ and $b$. Let $B$ be as in the conclusion of Theorem 1.1. To complete
the proof of the corollary, we will show that \( B \subset X_0 \cap X_1 \cap X_2 \). For this purpose, take an arbitrary point \( c \in B \). Since \( B \subset A_0 \cup A_1 \cup A_2 \), we may assume without loss of generality that \( c \in A_0 \). Since \( A_0 \subset X_1 \cap X_2 \), \( c \in X_1 \cap X_2 \). It remains to prove that \( c \in X_0 \). Suppose to the contrary that \( c \notin X_0 \). Since \( c \) is sheltered by \( A_1 \cup A_2 \subset X_0 \), \( c \) belongs to a bounded component of the complement of \( A_1 \cup A_2 \). By [7, (2.51) p. 107] or [3, Ch. X, §61, II, Theorem 5], there exists a simple closed curve \( J \subset A_1 \cup A_2 \) such that \( c \) belongs to the bounded component of the complement of \( J \). Since \( J \subset X_0 \) and \( X_0 \) is simply connected, the bounded component of the complement of \( J \) is contained in \( X_0 \). Hence, \( c \in X_0 \). \( \square \)

References