1995 ALGEBRA PRELIMINARY EXAMINATION

1. a. Prove that a subgroup of a cyclic group is cyclic.
   b. State and prove the classification theorem for cyclic groups.

2. a. State and prove the first Sylow theorem.
   b. Prove there is only one group (up to isomorphism) of order 65.

3. a. Prove for any nonempty set X, there exists an object free on X in the category of abelian groups.
   b. Is there a set X for which the additive group $\mathbf{Q}$ of rational numbers is free on X in the category of abelian groups? Explain.

4. Let G be a group.
   a. Define the *commutator subgroup* G' of G.
   b. Let N be a normal subgroup of G. Prove that G/N is abelian if and only if $G' \subseteq N$.

5. Let R be a ring.
   a. Define what is meant by a *maximal ideal* of R.
   b. Define what is meant by a *prime ideal* of R.
   c. If R is a principal ideal domain, prove that an ideal of R is maximal if and only if it is prime.

6. Prove that any finitely generated module M over a principal ideal domain is a direct sum of cyclics. (If you use the Stacked Bases Theorem, you need to prove it also for the sake of completeness.)

7. a. Define *artinian ring*.
   b. Prove or disprove that $\mathbf{Z}[x]$ is artinian.
   c. Define *noetherian ring*.
   d. Prove or disprove that $\mathbf{Z}[x]$ is noetherian.

8. Let R be a ring with 1.
   a. Prove that an R-module M is projective if and only if it is a direct summand of a free R-module, and give an example of a projective module that is not free.
   b. Outline the proof of the fact that any R-module M can be embedded in an injective R-module.

9. Let R be a ring with 1.

   a. Prove Nakayama's Lemma: If M is a nonzero finitely generated R-module, then $JM \neq M$. (J denotes the Jacobson radical of R, and JM denotes the submodule of M generated by all elements of the form jm, where $j \in J$ and $m \in M$.)
   b. State the structure theorem for artinian semisimple rings, and determine how many nonisomorphic artinian semisimple rings there are of order 2^{10}.