1996 ALGEBRA PRELIMINARY EXAMINATION

The examination is divided into two sections of six problems. In each section, you are required to do problem 6 and any four of the remaining five. If time permits you are encouraged to attempt all twelve problems.

I. Group Theory

1. Let $N \neq 1$ be a normal subgroup of A_5, the alternating group on 5 letters. Prove that N contains a 3-cycle.

2. (a) Give an example of a group that is not finitely generated.
(b) Using the minimal number of generators, exhibit generators and relations for the symmetric group S_3.
(c) Choose an appropriate free group F and exhibit a homomorphism φ from F onto S_3.

3. Prove or disprove: If A, B and C are abelian groups with $A \oplus B \cong A \oplus C$, then $B \cong C$.

4. With the exception of the trivial case where G is cyclic of prime order p, show that no group of prime power order can be simple.

5. It is easily verified that $p = 499$ is a prime.
   (a) Describe the structure of the multiplicative group G of the field $\mathbf{Z}/p\mathbf{Z}$ for $p = 499$.
   (b) Show that G does *not* contain an element of order 4.

6. (a) Describe (up to isomorphism) all the abelian groups of order 1996.
(b) Using generators and relations, describe (up to isomorphism) all the non-abelian groups G of order 1996 in each of the following two categories.
   (i) G has a cyclic group $\langle a \rangle$ for its Sylow 2-subgroup.
   (ii) G has the Klein 4-group, $\langle a \rangle \times \langle b \rangle$, for its Sylow 2-subgroup.

   For uniformity of notation in (i) and (ii), let c be an element of order 499 in G.

II. Ring Theory

Throughout this section, R is a ring with identity $1 \neq 0$ and all modules are assumed to be unitary.

1. Suppose R is commutative. Prove:
   (a) Every maximal ideal of R is prime.
   (b) If R is a principal ideal domain, every nonzero prime ideal of R is maximal.

2. Show that if R is a finite commutative ring such that $|R| = p$, a prime, then R is a field.

3. For any ring S, define

$$J(S) = \{s \in S : s \in M \text{ for every maximal ideal } M \text{ of } S\}.$$

Prove each of the following.

(a) $J(R)$ is an ideal of R and $J(R/J(R)) = 0$, the zero ideal of $R/J(R)$.

(b) If R is commutative, $a \in J(R)$ if and only if $1 + ra$ is a unit for every $r \in R$.

4. If N is a submodule of a left R-module M such that M/N is projective, show that $M = N \oplus P$ for some projective submodule P of M.

5. (a) Show $\mathbf{Z}/n\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Q} = 0$ for every integer $n \geq 1$.

(b) Give an example of a ring R, an exact sequence

$$0 \longrightarrow A \stackrel{\alpha}{\longrightarrow} B \stackrel{\beta}{\longrightarrow} C \longrightarrow 0$$

of left R-modules and a right R-module M such that the induced sequence

$$0 \longrightarrow M \otimes_R A \stackrel{1\otimes\alpha}{\longrightarrow} M \otimes_R B \stackrel{1\otimes\beta}{\longrightarrow} M \otimes_R C \longrightarrow 0$$

of abelian groups is *not* exact.

6. Suppose M is a nonzero left R-module. Recall that M is *simple* if the only submodules of M are 0 and M itself. Call M *indecomposable* if whenever $M = A \oplus B$ for submodules A and B, then either $A = 0$ or $B = 0$. Prove the following statements.

(a) If M is simple, $\mathrm{Hom}_R(M, M)$ is a division ring.

(b) If $\mathrm{Hom}_R(M, M)$ is a division ring, then M is indecomposable.

(c) Every simple module is indecomposable, but the additive group of rationals $\mathbf{Q}$ is an indecomposable $\mathbf{Z}$-module which is *not* simple.