April 1994 General Examination in Analysis (administered by J. B. Brown).

Work at least 8 problems.

1.a) Describe a function which is differentiable on [0,1] but not continuously differentiable there.

b) Describe a function which is of bounded variation on [0,1] but monotone on no subinterval of [0,1].

2.a) Define what it means to say that a subset M of [0,1] is (i) nowhere dense, (ii) first category, (iii) of Lebesgue measure zero, (iv) Lebesgue measureable.

b) Give an example (include details of construction) of a nowehere dense set which is of positive measure.

3. Prove that if $f\colon[0,1]\to R$ is Lebesgue measureable, and $g\colon R\to R$ is continuous, then $g\circ f$ (i.e. g[f]) is Lebesgue measurable.

(Hypothesis for 4-7) Let $f, f_1, f_2, \ldots$ be real valued functions which are measurable with respect to a σ-algebra A on a set Ω, and let μ be a (finite) measure with domain A.

4. Define what it means to say that (a) $\{f_n\}$ converges to f in measure μ, (b)$\{f_n\}$ converges to f uniformly, (c) $\{f_n\}$ converges to f almost everywhere (μ), (d) $\{f_n\}$ convergers to f in the $L^1(\mu)$ sense, (e) $\{f_n\}$ converges to f pointwise.

5. Prove that if $\{f_n\}$ converges to f in measure μ, then some subsequence of $\{f_n\}$ converges to f almost everywhere (μ).

6. Prove Egorov's theorem, i.e. that if $\{f_n\}$ converges almost everywhere (μ,) to f and $\varepsilon > 0$, then there is a set M such that $\mu(M^c) < \varepsilon$ and $\{f_n|M\}$ converges to $f|M$ uniformly.

7.a) State the "Lebesgue Dominated Convergence Theorem" (about moving "$\lim_{n\to\infty}$" inside or outside the integral sign).

b) Give an example of a sequence $\{f_n\}$ of continuous functions converging pointwise to a continuous function f on [0,1] such that

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx \quad\text{and}\quad \int_0^1 f(x)dx$$

both exists but are unequal.

8.a) Give two equivalent definitions (an "$\varepsilon-\delta-$partition" definition and another involving Lebesgue integrals) for what it means to say that a function $f\colon[0,1]\to R$ is absolutely continuous.

b) Give an example of a function $f\colon [0,1] \to R$ which is continuous and of bounded variation but is not absolutely continuous.

9.a) Define $L^p[0,1]$ and $L^p(R)$ for $0 < p \leq \infty$ { you can make the L^p-spaces collections of functions or collections of equivalence classes of functions, either way is OK }.

b) Assume that $f \in L^1[0,1]$. Which of the following must also belong to $L^1[0,1]$?

$$\text{(i)}\sqrt{|\text{f}|},\quad \text{(ii)}\text{f}^2,\quad \text{(iii)Arctan(f)}\quad \text{(give explanations).}$$

10.a) State Fubini's Theorem.

b) Give an example of a function $f\colon [0,1] \times [0,1] \to R$ such that

$$\int_0^1 \int_0^1 f(x,y)dxdy \text{ and } \int_0^1 \int_0^1 f(x,y)dydx$$

both exists but are unequal.