May 3, 1997 R. A. Zalik

Analysis Prelim

- The exam ends at 6 pm.
- The only measure used in this exam is Lebesgue measure, which is denoted by m.
- You may use Royden and one undergraduate level book. You may not use any notes.
- Please remain in the Math. Annex while you are taking the exam. Coffee, soft drinks, and lunch will be provided.
- Solve the first three questions and *either* part (a) *or* part (b) of each of the next seven questions. (If you solve both parts I will only give you credit for one.)
- Each question is worth 10 points, and the passing grade is 60 points.
- All the questions in this exam will be solved in class, beginning on Monday.
- The statement $p := q$ means that p equals q by definition.
- If anything is unclear, please come and see me. I will be in my office most of the time until the end of the exam.

1. Let $[a, b]$ be a closed bounded interval, assume that $f \in C[a, b]$ is of bounded variation and that $g \in C[a, b]$ is strictly increasing. For $a \le \alpha \le \beta \le b$, let $V(f, \alpha, \beta)$ denote the total variation of f on $[\alpha, \beta]$. Let $c \in [a, b]$ be arbitrary but fixed, and define $v(f, t)$ to equal $V(f, c, t)$ on $[c, b]$ and $-V(f, t, c)$ on $[a, c]$. Finally, let $q(t) := g(t) + v(f, t)$ and $h(t) := f[q^{-1}(t)]$. Prove that $h(t)$ is absolutely continuous on $[q(a), q(b)]$.

2. Let $y(t)$ be differentiable on $[0, \infty)$ and such that $y(0) = 1$ and, for every $x \in [0, \infty)$,

$$y'(x) = \frac{1}{x^2 + y^2(x)}$$

Prove that $\lim_{x\to\infty} y(x)$ exists and does not exceed $1 + \pi/2$.

3. (a) Prove that if f is a nonnegative real–valued measurable function on $[0, 1]$ and if $E_n := \{x : n - 1 \le f(x) < n\}$, then f is integrable if and only if $\sum_{n=1}^{\infty} n\, m(E_n) < \infty$.
(b) Prove that $\frac{\sin x}{x}$ is not Lebesgue integrable on $[0, \infty)$

4. (a) Prove that the Cantor ternary set has measure zero.
(b) Let f be absolutely continuous on $[c, d]$ and let g be absolutely continuous in $[a, b]$ with $c \le g \le d$. Prove that $f \circ g$ is absolutely continuous on $[a, b]$.

5. (a) Suppose that f and f_n, $n = 1, 2, \ldots$, are real valued measurable functions on a set E of finite measure, and, for every $\varepsilon > 0$,

$$E_n(\varepsilon) := \{x : |f_n(x) - f(x)| \ge \varepsilon\}, \quad n = 1, 2, \ldots.$$

Prove that the sequence $\langle f_n \rangle$ converges to f a. e. if and only if

$$\lim_{n\to\infty} m(E \cap \bigcup_{m=n}^{\infty} E_m(\varepsilon)) = 0$$

for every $\varepsilon > 0$.
(b) Suppose that f and f_n, $n = 1, 2, \ldots$, are real valued measurable functions on a set E of finite measure. Then $\langle f_n \rangle$ converges to f in measure if and only if every subsequence of $\langle f_n \rangle$ has in turn a subsequence that converges to f a. e.

6. (a) Prove that the partial sums S_n of the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{1 \,.\, 3 \ldots (2n-1)}{n!\, 2^n} x^n$$

of $f(x) := (1 - x)^{-1/2}$ converge to f in the norm of $L^1[-1, 1]$. (Hint: Use the Lebesgue Convergence Theorem.)
(b) Let Z denote the set of integers, let $f \in L^1[0, 1]$, and let $S := \{x \in [0, 1] : f(x) \in Z\}$. Prove that

$$\lim_{n\to\infty} \int_0^1 |\cos(\pi f(x))|^n \, dx = m(S).$$

(Hint: Use the Lebesgue Convergence Theorem.)

7. (a) Assume

$$\int_0^{\infty} |f_n'(x)|^2 \, dx \le M < \infty \quad \text{and} \quad |f_n(x)| \le x^{-1} \text{ on } (0, \infty)$$

for all natural numbers n. Prove that the sequence $\langle f_n \rangle$ contains a subsequence converging uniformly. (Hint: Use the Ascoli–Arzelá Theorem.)
(b) Prove that if $\langle g_n \rangle$ is a uniformly bounded sequence of real–valued monotone increasing functions (defined on $(-\infty, \infty)$), then there is a subsequence $\langle g_{n_k} \rangle$ and a monotone increasing function g, such that $\lim_{k\to\infty} g_{n_k}(x) = g(x)$ for every $x \in (-\infty, \infty)$.

8. (a) We say that a function $h : [a, b] \to R$ has the *intermediate value property* if for every $a \le c < d \le b$, if y is between $f(c)$ and $f(d)$ there is in (c, d) a point x such that $f(x) = y$. Prove that if a function h defined on a compact interval I is of bounded variation on I and also has the intermediate value property, then h is continuous on I. As a corollary, prove that if $f : [a, b] \to R$ is differentiable everywhere on $[a, b]$ and f' is of bounded variation on $[a, b]$ then f' is continuous on $[a, b]$.
(b) Show that the composition of two absolutely continuous functions can fail to be absolutely continuous. (Hint: Consider $f(x) := \sqrt{x}$, and $g(x)$ defined as follows: $g(0) := 0$. Divide $[0, 1]$ by the partition points 2^{-k}, $k \in N$, and thereby create open intervals $I_k := (2^{-k}, 2^{-k+1})$. Divide the interval I_k into 2^k equal subintervals $I_{k,m}$, $1 \le m \le 2^k$. On I_k let g be piecewise linear, nonnegative, and, starting at one end of I_k, let g take on alternately the values 0 and 2^{-2k} at the endpoints of the intervals $I_{k,m}$. Don't forget to prove that f and g are absolutely continuous.)

9. (a) Let $1 \le p < \infty$. Prove that every Cauchy sequence in $L^\infty[0, 1]$ converges both pointwise (a. e.) and in $L^p[0, 1]$.
(b) Prove that if f is integrable on the interval $[0, 1]$ and $\int_0^1 f(x)\,dx \ne 0$, then there is a measurable function g such that

$$\int_0^1 |f(x)g(x)|\,dx < \infty \quad \text{and} \quad \int_0^1 |f(x)||g(x)|^2\,dx = \infty.$$

10. (a) Let Y be a closed subspace of a Banach space X. Prove that if $x \in X \setminus Y$ and $a > 0$, then the linear span of $Y \cup \{x\}$ contains an element z such that $\|z\| \le 1$ and $d(z, Y) := \inf\{\|z - y\| : y \in Y\} > 1 - a$.
(b) Let f be a linear functional on a normed linear space. Assume that the kernel of f is closed. Is f continuous? Prove or disprove.