TOPOLOGY PRELIMINARY EXAMINATION
Wednesday, June 14, 1995

Do problems 1 through 4, plus your choice of any four problems in the ones numbered 5 through 12. All problems are weighted equally. Do each problem on a separate page. Include a cover page which lists the problems you have chosen.

Problem 1. Let $f : X \to X$ be a continuous map on a Hausdorff space. Show that the set of fixed points of f is closed.

Problem 2. Suppose that H and K are compact spaces. Show that $H \times K$ is compact.

Problem 3. Prove the Baire category theorem for compact Hausdorff spaces.

Problem 4. Let C_n be compact subsets of a topological space, with $C_{n+1} \subseteq C_n$, and let $C = \bigcap_{i=1}^{\infty} C_i$. Prove that C is nonempty and compact.

Problem 5. Let $\mathcal{G}$ be a collection of connected subsets of a topological space, and suppose that there is a point p such that p is a point or a limit point of each element of $\mathcal{G}$, and is an element of at least one element of $\mathcal{G}$. Show that the union of the elements of $\mathcal{G}$ is connected.

Problem 6. Let X be a compact Hausdorff space, and let $f : X \to X$ be continuous. Let $S = \{x \in X^\omega : \text{For all } n \in \omega,\ x_n = f(x_{n+1})\}$, where ω is the set of all nonnegative integers. Prove that S is a compact subset of X^ω, where X^ω has the usual product topology.

Problem 7. Prove that every separable metric space has a countable basis.

Problem 8. Show that if S is a well ordered set which has a last element, then S with the order topology is compact.

Problem 9. Let M be a metric continuum. Show that if x and y are two points of M, then there is a subcontinuum of M which is irreducible from the point x to the point y. [**Definition:** a *continuum* is a compact connected space. **Definition:** A continuum X is *irreducible* from the point x to the point y iff X contains x and y, but no proper subcontinuum of X contains both x and y.]

Problem 10. Give an example of a regular Hausdorff space which is not normal, and show why.

Problem 11. Suppose that X is a topological space. Let α be a path in X from the point a to the point b, and let β and γ be paths in X from the point b to the point c. Prove that $\alpha * \beta$ and $\alpha * \gamma$ are path homotopic if and only if β and γ are path homotopic. [You may use the fact that the path homotopy relation is an equivalence relation, but otherwise any claim that two paths are path homotopic must be justified by giving the relevant homotopy.]

Problem 12. Let X consist of the usual Euclidean plane with the origin removed. Find a universal covering space for X, along with the corresponding covering map. Then use this covering space to prove that X is not simply connected.