ON A NEW SHAPE INVARIANT

by

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1. Introduction

The n-dimensional homology group \( H_n(X, \mathcal{G}) \) of a compactum \( X \) over an Abelian group \( \mathcal{G} \) is understood here always in the sense of Vietoris (see, for instance [3], p. 36). Thus the elements of a such group are homology classes \( (y) \) in \( X \), where \( y \) is a true n-dimensional cycle in \( X \) with coefficients belonging to \( \mathcal{G} \). For simplicity, we shall write "cycle" instead of "true cycle" and we may assume that \( X \) is a subset of the Hilbert cube \( Q \).

Let \( X' \) be another compactum (lying in \( Q \)) and let \( f: X \rightarrow X' \) be a fundamental sequence (concerning this notion, and also other notions of the theory of shape, see [3]). It is well known ([3], p. 70) that \( f \) induces a homomorphism

\[
\tilde{f}_n : H_n(X, \mathcal{G}) \rightarrow H_n(X', \mathcal{G})
\]

assigning covariantly to every homology class \( (\gamma) \in H_n(X, \mathcal{G}) \) a homology class \( \tilde{f}_n ((\gamma)) \in H_n(X', \mathcal{G}) \).

By a power domain \((Z, a_k)\) one understands (compare [3], p. 91) a system consisting of a set \( Z \) and of a family of functions \( a_k: Z \rightarrow Z \) assigned to indices \( k = 0, \pm 1, \pm 2, \cdots \) and such that

\[
a_1(z) = z \quad \text{and} \quad a_k a_m(z) = a_{km}(z) \quad \text{for every} \quad z \in Z \quad \text{and} \quad k, m = 0, \pm 1, \pm 2, \cdots
\]

By a homomorphism of \((Z, a_k)\) into another power domain \((Z', a_k')\) one understands a function \( \phi: Z \rightarrow Z' \) such that

\[
\phi a_k(z) = a_k' \phi(z) \quad \text{for every} \quad z \in Z \quad \text{and} \quad k = 0, \pm 1, \pm 2, \cdots
\]

Two power domains \((Z, a_k), (Z', a_k')\) are said to be isomorphic if there exists a one-to-one homomorphism \( \phi \) of \((Z, a_k)\) onto \((Z', a_k')\). It is clear that then the function \( \psi = \phi^{-1} : Z' \rightarrow Z \) is also a homomorphism. If we assume only that there exist two
homomorphisms \( \phi \) of \((z, \alpha_k)\) into \((z', \alpha'_k)\) and \( \psi \) of \((z', \alpha'_k)\) into \((z, \alpha_k)\) such that \( \phi \psi \) is the identity, then we say that the power domain \((z, \alpha_k)\) \(r\)-dominates the power domain \((z', \alpha'_k)\).

2. Cancellable Cycles

Let \( \Omega \) be a family of \(n\)-dimensional cycles in a compactum \(X\) over an Abelian group \(\mathcal{A}\) and let \(m\) be a non-negative integer. An \(m\)-dimensional cycle \(y\) in \(X\) over \(\mathcal{A}\) is said to be cancellable rel. \(\Omega\) provided there exists a fundamental sequence \(f: X \to X\) such that:

\[
\begin{align*}
(2.1) & \quad f^{-n} ((\omega)) = (\omega) \text{ for every cycle } \omega \in \Omega, \\
(2.2) & \quad f^{-m} ((\gamma)) = 0.
\end{align*}
\]

Then we say that \(f\) realizes the cancellation of \(y\) rel. \(\Omega\).

It is clear that the cancellability of \(y\) rel. \(\Omega\) depends only on the homology class \((\gamma)\) of \(y\) and on the collection \((\Omega)\) of the homology classes of cycles belonging to \(\Omega\). Consequently we may speak about homology classes cancellable rel. \((\Omega)\).

Observe that if \(y\) is an \(m\)-dimensional cycle in \(X\) over \(\mathcal{A}\), cancellable rel. \(\Omega\), then for every integer \(k\) the cycle \(k \cdot y\) is also cancellable rel. \(\Omega\). Consequently the collection \(Z\) of all \(m\)-dimensional homology classes in \(X\) over \(\mathcal{A}\), cancellable rel. \((\Omega)\), is a power domain \((Z, \alpha_k)\), where the function \(\alpha_k\) is defined by the formula \(\alpha_k ((\gamma)) = k(\gamma)\). Let us denote this power domain by \(\Gamma_m (X, \mathcal{A}, (\Omega))\). In special case when \((\Omega) = H_n (X, \mathcal{A})\), we shall write \(\Gamma_m (X, \mathcal{A}, n)\) instead of \(\Gamma_m (X, \mathcal{A}, (\Omega))\). In the case when \(\mathcal{A} = \mathbb{N}\) is the group of integers, we shall write \(\Gamma_m (X, n)\) instead of \(\Gamma_m (X, \mathbb{N}, n)\).

(2.3) Problem. Is it true that for every compactum \(X\), for every \(m \neq n\), for every Abelian group \(\mathcal{A}\), and for every \((\Omega) \subset H_n (X, \mathcal{A})\) the power domain \(\Gamma_m (X, \mathcal{A}, (\Omega))\) is a subgroup of the group \(H_m (X, \mathcal{A})\)?
3. Examples

In order to illustrate the sense of the cancellability, let us give some simple examples:

(3.1) Example. Let \( X = X_1 \cup X_2 \), where \( X_1, X_2 \) are compacta and \( X_1 \cap X_2 \in \text{AR} \). Then every \( m \)-dimensional cycle \( \gamma \) in \( X_1 \) over any Abelian group \( \mathcal{A} \) is cancellable in \( X \) rel. each \( n \)-dimensional cycle \( \omega \) in \( X_2 \) over \( \mathcal{A} \).

In fact, since \( X_1 \cap X_2 \in \text{AR} \), there is a retraction \( r: X_1 \to X_1 \cap X_2 \). Setting \( f(x) = r(x) \) for \( x \in X_1 \) and \( f(x) = x \) for \( x \in X_2 \), we get a retraction \( f: X \to X_2 \). It is clear that \( f \) realizes the cancellation of \( \gamma \) rel. the family of all \( n \)-dimensional cycles in \( X_2 \) over \( \mathcal{A} \).

(3.2) Example. Let \( X = X_1 \cup X_2 \), where \( X_1 \) and \( X_2 \) are compacta, \( m_1 = \dim X_1 \) and \( X_1 \cap X_2 \in \text{AR} \). If \( m \leq m_1 < n \), then

\[
\Gamma_m(X, \mathcal{A}, n) 
\leq H_m(X_1, \mathcal{A}).
\]

In order to show this, consider an \( n \)-dimensional cycle \( \omega \) in \( X \) over \( \mathcal{A} \). Then there exists a cycle \( \omega' \in (\omega) \) of the form

\[
\omega' = \kappa_1 - \kappa_2,
\]

where \( \kappa_v \) is an infinite \( n \)-dimensional chain in \( X_v \) over \( \mathcal{A} \), for \( v = 1, 2 \). Since \( X_1 \cap X_2 \in \text{AR} \), there exists in \( X_1 \cap X_2 \) an infinite \( n \)-dimensional chain \( \mu \) over \( \mathcal{A} \) such that \( \partial \mu = \partial \kappa_1 = \partial \kappa_2 \). Then \( \kappa_1 - \mu \) is an infinite \( n \)-dimensional chain in \( X_1 \) over \( \mathcal{A} \). Since \( \dim X_1 = m_1 < n \), there exists in \( X_1 \) an infinite \((n+1)\)-dimensional chain over \( \mathcal{A} \) having \( \kappa_1 - \mu \) as its boundary. It follows that

\[
\omega' = (\kappa_1 - \mu) - (\kappa_2 - \mu) \sim \mu - \kappa_2 \text{ in } X.
\]

But \( \mu - \kappa_2 \) lies in \( X_2 \) and we infer by Example (3.1) that each \( n \)-dimensional cycle \( \gamma \) over \( \mathcal{A} \) is cancellable rel. \( \mu - \kappa_2 \), hence also cancellable relatively \( \omega \) in \( X \).

(3.3) Example. Let \( T \) be the surface of a torus. We may
consider $T$ as the Cartesian product of two circles $S^1$, that is every point $x \in T$ may be represented in the form $(x_1, x_2)$, where $x_1, x_2 \in S^1$.

Consider a point $a \in S^1$ and let $X_1$ denote the circle consisting of all points of the form $(a, x_2)$. Let $\gamma$ be a 1-dimensional cycle in $X_1$ over the group of integers $\mathbb{Z}$, such that the homology class $(\gamma)$ is a generator of the Betti group $H_1(X_1)$. Moreover, let $\omega$ be a 2-dimensional cycle in $T$ over $\mathbb{Z}$ such that $(\omega)$ be a generator of the cyclic infinite Betti group $H_2(T)$. Let us show that $\gamma$ is not cancellable rel. $\omega$ in $T$.

Consider a fundamental sequence $f : T \to T$ such that

$$(3.4) \quad f_2((\omega)) = (\omega).$$

Since $T \in \text{ANR}$, the fundamental sequence $f$ is generated by a map $f : T \to T$ and we infer that $f(\omega)$ is a 2-dimensional cycle homologous to $\omega$ in $T$. Then $f$ maps the oriented circle $X_1$ onto a loop in $T$.

Suppose that this loop is homologous to zero in $T$. Since the 1-dimensional fundamental group of $T$ is Abelian (see, for instance [8], p. 149) this loop is homotopic in $T$ to a constant and we infer that the map $f$ is homotopic to a map $f'$ by which the circle $X_1$ passes onto one point. It follows easily that $f'$ may be represented in the form $\psi \phi$, where $\phi$ maps $T$ onto a set $A$ which is the union of a 2-dimensional sphere $S^2$ and of one of its diameters. But it is known (see [1], p. 526) that every map of $S^2$ into $T$ is homotopic to a constant. Consequently the map $\psi$ is homotopic to a map of $A$ onto a 1-dimensional subset of $T$ and we infer that $\psi(\omega) \not\sim \omega$ in $T$, which contradicts (3.4). Hence the cycle $\gamma$ is not cancellable rel. $\omega$ in $T$.

Remark. The fact considered in Example (3.3) is a direct consequence of the following theorem, due to A. Bojanowska [2]:

$$(3.5) \text{Theorem. Let } M^m \text{ be a closed, compact and connected}$$
n-dimensional orientable manifold and let $\omega$ be an $n$-dimensional cycle in $M_n$ such that $(\omega)$ is a generator of the Betti group $H_n(M_n)$. Then no $m$-dimensional cycle in $M_n$ over any Abelian group $\mathcal{A}$ is cancellable rel. $\omega$ in $M_n$.

4. Shape Invariance of

\[ \Gamma_m(X, \mathcal{A}, (\Omega)). \] Let us prove the following

(4.1) Theorem. Let $X, X'$ be compacta, $\mathcal{A}$ be an Abelian group, $\varphi: X \to X'$ be a fundamental sequence and $\Omega$ be a collection of $n$-dimensional cycles in $X$ over $\mathcal{A}$. Then:

(I) If there exists a fundamental sequence $\hat{\varphi}: X' \to X$ such that $\hat{\varphi} \varphi = i_X$, then the homomorphism $\varphi^m / \Gamma_m(X, \mathcal{A}, (\Omega))$ is a right inverse of the homomorphism $\hat{\varphi}^m / \Gamma_m(X', \mathcal{A}, \varphi_n((\Omega)))$.

(II) If there exists a fundamental sequence $\hat{\varphi}: X' \to X$ such that $\hat{\varphi} \varphi = i_X$ and $\varphi \hat{\varphi} = i_{X'}$, then $\varphi^m / \Gamma_m(X, \mathcal{A}, (\Omega))$ is an isomorphism.

Proof. Assume that there exists a fundamental sequence $\hat{\varphi}: X' \to X$ such that $\hat{\varphi} \varphi = i_X$. The fundamental sequences $\varphi$ and $\hat{\varphi}$ induce homomorphisms:

\[ \varphi_m : H_m(X, \mathcal{A}) \to H_m(X', \mathcal{A}), \quad \varphi_n : H_n(X, \mathcal{A}) \to H_n(X', \mathcal{A}), \]
\[ \hat{\varphi}_m : H_m(X', \mathcal{A}) \to H_m(X, \mathcal{A}), \quad \hat{\varphi}_n : H_n(X', \mathcal{A}) \to H_n(X, \mathcal{A}) \]

such that

\[ \hat{\varphi}_m \varphi_m = i_{H_m(X, \mathcal{A})}, \quad \hat{\varphi}_n \varphi_n = i_{H_n(X, \mathcal{A})}. \]

Consider an $m$-dimensional cycle $\gamma$ in $X$ over $\mathcal{A}$ such that $(\gamma) \in \Gamma_m(X, \mathcal{A}, (\Omega))$ and let $\omega$ be a cycle belonging to $\Omega$. Setting

\[ (\gamma') = \varphi_m((\gamma)) \quad \text{and} \quad (\omega') = \varphi_n((\omega)) \in \varphi_n((\Omega)), \]

we infer by $\hat{\varphi} \varphi = i_X$ that

\[ \hat{\varphi}_m((\gamma')) = \varphi_m \varphi((\gamma)) \quad \text{and} \quad \hat{\varphi}_n((\omega')) = \varphi_n \varphi((\omega)) = (\omega). \]

Since $\gamma \in \Gamma_m(X, \mathcal{A}, \Omega)$ and $\omega \in \Omega$, there exists a fundamental
sequence \( f: X \to X \) such that the homomorphisms \( f_m \) and \( f_n \) satisfy the conditions:

\[
(4.4) \quad f_m((\gamma)) = 0 \quad \text{and} \quad f_n((\omega)) = (\omega), \quad \text{for every} \quad \omega \in \Omega.
\]

Setting

\[
\hat{f} = g \circ f \circ \hat{g}: X' + X',
\]

we infer by (4.2), (4.3) and (4.4) that

\[
\hat{f}_m((\gamma')) = g_m \circ f_m \circ \hat{g}_m \circ g_m((\gamma)) = g_m \circ f_m ((\gamma)) = 0
\]

and

\[
\hat{f}_n((\omega')) = g_n \circ f_n \circ \hat{g}_n \circ g_n((\omega)) = g_n \circ f_n ((\omega)) = g_n((\omega)) = (\omega').
\]

Thus we have shown that the fundamental sequence \( \hat{f} \) realizes the cancellability of the cycle \( \gamma' \) relatively the \( n \)-dimensional cycle \( \omega' \). Hence the homomorphism \( g_m \) assigns to every element \((\gamma)\) of \( \Gamma_m(X, G, (\Omega)) \) an element \((\gamma')\) of \( \Gamma_m(X', G, g_n((\Omega))) \). Moreover the homomorphism \( \hat{g}_m \) assigns to \((\gamma') = g_m((\gamma))\) the element \((\gamma)\) of \( \Gamma_m(X, G, (\Omega)) \). It follows that the power domain \( \Gamma_m(X', G, g_n((\Omega))) \) \( r \)-dominates the power domain \( \Gamma_m(X, G, (\Omega)) \) and the proof of proposition (I) is finished.

If the hypotheses of (II) are satisfied, then one shows in the same way that the homomorphism \( \hat{g}_m \) assigns to each element \((\gamma')\) of \( \Gamma_m(X', G, g_n((\Omega))) \) an element \((\gamma)\) of \( \Gamma_m(X, G, (\Omega)) \) and that both relations

\[
\hat{g}_m \circ g_m = i_{H_m(X, G)} \quad \text{and} \quad g_m \circ \hat{g}_m = i_{H_m(X', G)}
\]

hold true. It follows that the power domain \( \Gamma_m(X, G, (\Omega)) \) is isomorphic to the power domain \( \Gamma_m(X', G, g_n((\Omega))) \). Thus the proof of (II) is finished and Theorem (4.1) is established.

(4.5) Corollary. If \( \text{Sh}(X) = \text{Sh}(X') \), then the power domain \( \Gamma_m(X, G, n) \) is isomorphic to the power domain \( \Gamma_m(X', G, n) \).

In fact, the relation \( \text{Sh}(X) = \text{Sh}(X') \) implies that there exist two fundamental sequences \( g:X \to X' \) and \( \hat{g}:X' \to X \) satisfying the conditions:
\[ q \circ q = i_X \quad \text{and} \quad q \circ q = i_{X'} \, . \]

Then the induced homomorphism \( q_n : H_n(X, \mathcal{A}) \to H_n(X', \mathcal{A'}) \) is an isomorphism. It follows that if \( \Omega \) denotes the collection of all \( n \)-dimensional cycles in \( X \) over \( \mathcal{A} \) and \( \Omega' \) denotes the collection of all \( n \)-dimensional cycles in \( X' \) over \( \mathcal{A} \), then \( q_n((\Omega)) = (\Omega') \) and consequently:

\[ \Gamma_m(X, \mathcal{A}, (\Omega)) = \Gamma_m(X, \mathcal{A}, n) \quad \text{and} \quad \Gamma_m(X', \mathcal{A}, q_n((\Omega))) = \Gamma_m(X', \mathcal{A}, n). \]

It suffices to apply Theorem (4.1), (II) in order to infer that \( \Gamma_m(X, \mathcal{A}, n) \) is isomorphic to \( \Gamma_m(X', \mathcal{A}, n) \).

Thus the power domain \( \Gamma_m(X, \mathcal{A}, n) \) is a shape invariant of \( X \).

5. Application

The shape invariance of \( \Gamma_m(X, \mathcal{A}, n) \) allows us to prove the following

(5.1) Theorem. Let \( X \) be a continuum satisfying the following conditions:

1° \( X \) is movable,
2° The number \( n = Fd(X) \) is finite,
3° The Betti group \( H_n(X) \) is a cyclic infinite group,
4° \( X \) is approximatively \( 1 \)-connected,
5° \( \Gamma_m(X, n) = 0 \), for \( m < n \).

Then for every point \( a \in X \), the shape \( Sh(X, a) \) is simple.

We shall use in the proof of this theorem the following, well-known proposition:

(5.2) If \( X \) is a movable, approximatively \( 1 \)-connected continuum with \( Fd(X) \leq n \) and if the Betti groups \( H_m(X) \) vanish for \( m = 1, 2, \ldots, n \), then \( Sh(X) \) is trivial.

In order to see it, consider a point \( a \in X \). It follows by the well-known modified Hurewicz theorem (due to K. Kuperberg [6], p. 26) that the fundamental groups \( \pi_m(X, a) \) vanish for
m = 1, 2, ..., n. Using the modified theorem of Whitehead, transferred into theory of shape by M. Moszyńska ([7], p. 260), see also J. Keesling ([5], p. 248), we infer that setting
\[ f_k(x) = a \] for every point \( x \in Q \) and for \( k = 1, 2, \cdots \),
one obtains a fundamental sequence \( \mathbf{f} = \{f_k, X, (a)\} \) which is a fundamental equivalence. Hence \( \text{Sh}(X) = \text{Sh}(a) \), i.e. \( \text{Sh}(X) \) is trivial.

**Proof of Theorem (5.1).** Otherwise there would exist two continua \( X_1, X_2 \) with non-trivial shapes such that
\[ X_1 \cap X_2 = (a) \quad \text{and} \quad \text{Sh}(X) = \text{Sh}(X_1 \cup X_2). \]
It is clear that \((X_1, a)\) and \((X_2, a)\) are retracts of \((X, a)\) and we infer by 1°, 2° and 4° that \( X_1, X_2 \) are movable, \( Fd(X_1), Fd(X_2) < n \) and that \( X_1 \) and \( X_2 \) are approximatively 1-connected. Moreover 3° implies that at least one of the groups \( H_n(X_1), H_n(X_2) \) vanishes. We may assume that \( H_n(X_1) = 0 \).

It is clear that every cycle lying in \( X_1 \) is cancellable rel. the family of all cycles lying in \( X_2 \) and that every \( n \)-dimensional cycle \( \omega \), with integers as coefficients, lying in \( X \) is homologous (in \( X \)) to the sum \( \omega_1 + \omega_2 \) of two cycles lying in \( X_1 \) and \( X_2 \) respectively. Since \( H_n(X_1) = 0 \), we infer that \( \omega_1 \sim 0 \) in \( X_1 \) and consequently \( \omega \) is homologous to a cycle lying in \( X_2 \). It follows that every \( m \)-dimensional cycle \( \gamma \) lying in \( X_1 \), with \( m < n \), is cancellable relatively every \( n \)-dimensional cycle lying in \( X \) and we infer by 5° that \( \gamma \sim 0 \) in \( X \). Since \( X_1 \) is a retract of \( X \), it is also \( \gamma \sim 0 \) in \( X_1 \). Thus \( H_m(X_1) = 0 \) for \( m = 1, 2, \cdots, n \). It follows by (5.2) that \( \text{Sh}(X_1) \) is trivial, contrary to our supposition. Thus the proof of Theorem (5.1) is finished.

Using Theorems (3.5) and (4.1), we get the following

**(5.3) Corollary.** If \( M_n \) is a closed, compact, connected, 1-connected \( n \)-dimensional manifold and if \( a \in M_n \), then \( \text{Sh}(M_n, a) \)
is simple.

In order to obtain this corollary from Theorem (5.1), let us notice that $X = M_n$ satisfies the conditions $1^\circ$, $2^\circ$, $3^\circ$, $4^\circ$. Also condition $5^\circ$ is satisfied, because of Theorem (3.5).

The problem if the hypothesis that $M_n$ be 1-connected is essential for Corollary (5.3) remains open.¹ Let us only mention, that in the case $n=2$ this hypothesis may be omitted, as it has been shown recently by A. Kadlof [4].

References


¹R. Sher has informed me, in an oral communication, that the hypothesis that the manifold $M_n$ is 1-connected is superfluous. This result will appear in a joint paper of R. Sher and J. Hollingsworth.