ON MONOTONE RETRACTS,
ACCESSIBILITY, AND SMOOTHNESS IN CONTINUA

by

G. R. GORDH, JR. AND LEWIS LUM
ON MONOTONE RETRACTS, ACCESSIBILITY, AND SMOOTHNESS IN CONTINUA

G. R. Gordh, Jr. and Lewis Lum

1. Introduction

Consider the following conditions which a continuum M may satisfy.

(*) Each subcontinuum of M is a monotone retract of M.

(**) (Pointed version) Each subcontinuum of M which contains a fixed point p is a monotone retract of M.

It is easy to verify that dendrites satisfy both conditions (see [10], Theorem 2.1). The second author has proved that if M is a dendroid, then each of (*) and (**) implies that M is a dendrite ([10], Theorem 2.3, and [12], Theorem 3). More recently, the authors have obtained the same conclusion for arbitrary metric continua satisfying (*), and for arcwise connected metric continua satisfying (**) [6].

In particular, (*) and (**) are equivalent for arcwise connected continua. However, they are not equivalent in general since the familiar "sin 1/x curve" satisfies (**).

Thus it is natural to ask for a characterization of continua satisfying (**). The main purpose of this paper is to provide such a characterization.

Theorem. A continuum M satisfies (**) if and only if

(a) M is smooth at p, and

(b) for each subcontinuum N of M containing p, N is accessible and the components of M - N form a null family.

In this result "smoothness" refers to the concept introduced by the first author in [4]. A more general definition
of "smoothness" has been studied by T. Maćkowiak [14].

It is also shown that a metric continuum $M$ satisfying (**) becomes a dendrite under the canonical monotone decomposition $\mathcal{D}$ of smooth continua defined in [4]. Thus condition (*) is recovered in the decomposition space $M/\mathcal{D}$.

2. Definitions and Preliminary Remarks

A continuum is a compact connected Hausdorff space. The reader is referred to [7] for basic properties of continua and undefined terms.

A subcontinuum $N$ of a continuum $M$ is called a monotone retract of $M$ if there exists a mapping $r: M \rightarrow N$ which is both monotone and a retraction.

Let $X$ be a subset of a continuum $M$. A point $x \in X$ is said to be accessible from a point $y \in M - X$ if there exists a subcontinuum $H$ such that $y \in H$ and $H \cap X = \{x\}$. If some point of $X$ is accessible from some point of $M - X$, then $X$ is called accessible.

A collection $\mathcal{C}$ of subsets of a continuum $M$ will be called a null family if each convergent net $C_n$ of elements of $\mathcal{C}$ which is not eventually constant has a degenerate limit.

A continuum $M$ is irreducible from the point $p$ to the point $q$ if no proper subcontinuum of $M$ contains $p$ and $q$. If, in addition, no proper connected subset of $M$ contains $p$ and $q$, then $M$ is called an arc (sometimes generalized arc or ordered continuum).

The continuum $M$ is hereditarily unicoherent at $p$ if for each pair of subcontinua $H$ and $K$ containing $p$, $H \cap K$ is connected; or equivalently, if for each $q$ in $M - \{p\}$, there is a unique subcontinuum, denoted by $pq$, which is irreducible from $p$ to $q$. If $M$ is hereditarily unicoherent at $p$ and for each convergent net $q_n$, $\lim q_n = q$ implies that the net of subcontinua $pq_n$ converges to $pq$, then $M$ is said to be smooth at $p$ [4].
A tree (dendrite) is a locally connected, hereditarily unicoherent (metric) continuum. A generalized tree (smooth dendroid) is an arcwise connected, hereditarily unicoherent, smooth (metric) continuum.

Let $M$ be a continuum which is hereditarily unicoherent at the point $p$. The weak cutpoint order on $M$ with respect to $p$ will be denoted by $\preceq$ (i.e., $x \preceq y$ if $px \subseteq py$). For each $x \in M$ the set $D(x) = \{y \in M: py = px\}$ is the level set of $x$ relative to $\preceq$. The collection $\mathcal{D}$ of all level sets forms a decomposition (not necessarily upper semicontinuous) of $M$. Let $\phi: M \to M/\mathcal{D}$ denote the natural mapping where $M/\mathcal{D}$ is given the quotient topology. Observe that for any subcontinuum $N$ of $M$ which contains $p$, $\phi^{-1}(\phi(N)) = N$. We now list, for reference, some of the basic facts concerning the decomposition $\mathcal{D}$.

(i) For each $x \in M$, $D(x)$ is connected (see [9], Theorem 3, p. 210 for metric continua, and [2], Theorem 1.2 for the general case).
(ii) For each $x \in M$, $D(x)$ has void interior in $px$ (see [7], Theorem 3-44).
(iii) If $M$ is smooth at $p$, then $\mathcal{D}$ is a monotone upper semicontinuous decomposition and $M/\mathcal{D}$ is a generalized tree which is smooth at $D(p)$ (see [4], Theorem 5.2 and Theorem 4.1).
(iv) If $M/\mathcal{D}$ is a continuum which is smooth at $D(p)$, then $M$ is smooth at $p$ (see [13], Theorem 3.1 for metric continua, and [11], Theorem 6.3 for the general case).

3. The Main Results

Throughout this section $M$ will denote a continuum containing a fixed point $p$.

We shall prove

Theorem 1. Each subcontinuum of $M$ which contains $p$ is a
monotone retract of $M$ (i.e., $M$ satisfies (**)) if and only if

(a) $M$ is smooth at $p$, and

(b) for each subcontinuum $N$ of $M$ containing $p$, $N$ is accessible and the components of $M - N$ form a null family.

Furthermore, if (**) holds, then $M/\mathcal{D}$ is a tree.

We shall need several lemmas.

**Lemma 1.** Let $M$ be hereditarily unicoherent at $p$, and let $N$ and $P$ be subcontinua of $M$ such that $p \in N \subseteq P$. If $r: M \to N$ is a monotone retraction, then $r|P$ is a monotone retraction.

**Proof.** It suffices to show that $r^{-1}(x) \cap P$ is connected for each $x \in N$. If not, there exist disjoint closed sets $A$ and $B$ such that $r^{-1}(x) \cap P = A \cup B$ and $x \in A$. But this contradicts hereditary unicoherence at $p$ since $(r^{-1}(x) \cup N) \cap P = (A \cup N) \cup B$ and $(A \cup N) \cap B = \emptyset$.

**Lemma 2.** Let $M$ be irreducible from $p$ to $q$. If each subcontinuum of $M$ which contains $p$ is a monotone retract of $M$, then $M$ is smooth at $p$.

**Proof.** According to the Lemma of [6], $M$ is hereditarily unicoherent at $p$. Thus, by (iv) of Section 2, it suffices to show that $M/\mathcal{D}$ is a continuum which is smooth at $D(p)$. We begin by showing that $D(z)$ is closed for each $z$ in $M$. First suppose that $x$ and $y$ belong to $\operatorname{cl}(D(z)) - D(z)$. By the hypothesis and Lemma 1, there is a monotone retraction $r: pz \to px \cup py$. By irreducibility $pz = (px \cup py) \cup r^{-1}(r(z))$. Thus $\{x, y\} \subseteq r(D(z)) = r(z)$ and $x = y$. In particular, $\operatorname{cl}(D(z)) - D(z) = \{x\}$. Since $D(z)$ is connected (by (i) of Section 2) and $pz$ is irreducible, $pz = px \cup D(z)$. But this implies that $D(z)$ has nonvoid interior in $pz$, contradicting (ii) of Section 2. Thus $D(z)$ is closed. We now show that each element $D(z)$ of $\mathcal{D}$ distinct from $D(p)$ and $D(q)$ separates $D(p)$
from \(D(q)\) in \(M/\mathcal{D}\). Let \(z \in M - (D(p) \cup D(q))\) and let \(r:M \to pz\) be a monotone retraction. By irreducibility \(M = pz \cup r^{-1}(r(q))\), and \(r(q) \in D(z)\). Since \(r(q)\) separates \(p\) from \(q\) in \(M\), \(D(z)\) separates \(D(p)\) from \(D(q)\) in \(M/\mathcal{D}\). It follows that \(M/\mathcal{D}\) is an arc (e.g., [3], Theorem 2.1). Consequently \(M/\mathcal{D}\) is a continuum which is smooth at \(D(p)\).

A quite different (and somewhat longer) proof of Lemma 2 can be obtained by applying the characterization of smoothness for irreducible continua given by J. J. Charatonik in [1]. Example 2 in Section 4 shows that the converse of Lemma 2 is false.

**Lemma 3.** Let \(M\) be hereditarily unicoherent at \(p\) and assume that \(M/\mathcal{D}\) is a tree. Let \(N\) be a subcontinuum of \(M\) containing \(p\) and let \(C\) be a component of \(M - N\). Then

(a) \(C\) is open and continuumwise connected.

(b) At most one point of \(N\) is accessible from any point of \(C\).

(c) If \(r:M \to N\) is a monotone retraction, then \(r(C)\) is degenerate.

**Proof.** Note that \(M\) is smooth at \(p\) by (iv) of Section 2.

(a) Using the facts that \(\phi:M \to M/\mathcal{D}\) is monotone and \(\phi^{-1}(\phi(N)) = N\) (see (iii) of Section 2), it is easy to verify that \(\phi^{-1}(\phi(C)) = C\). It follows that \(\phi(C)\) is a component of \(M/\mathcal{D} - \phi(N)\). As a component of an open subset of a tree, \(\phi(C)\) is open and arcwise connected. Thus \(C = \phi^{-1}(\phi(C))\) is open and continuumwise connected.

(b) Suppose that \(x\) and \(y\) are distinct points of \(N\) which are accessible from points in \(C\). Then there exist subcontinua \(X\) and \(Y\) of \(M\) such that \(X \cap C \neq \emptyset \neq Y \cap C\), \(X \cap N = \{x\}\), and \(Y \cap N = \{y\}\). Applying (a), there exists a subcontinuum \(K \subseteq C\) such that \(X \cap K \neq \emptyset \neq Y \cap K\).
But then \((N \cup X \cup K) \cap (N \cup Y \cup K) = N \cup (K \cup (X \cap Y))\)
which is a separation, contradicting hereditary uni-
coherence at \(p\).

(c) If \(x, y \in r(C)\), then \(r^{-1}(x) \cap C \neq \emptyset \neq r^{-1}(y) \cap C\). Thus
\(x = y\) by (b).

We shall need the notion of aposyndesis due to F. B. Jones
(see [8] for a discussion of the history of this concept). A
continuum \(M\) is said to be aposyndetic at \(x\) with respect to \(y\)
if there exists a subcontinuum \(K\) of \(M\) such that \(x \in \text{int}(K) \subseteq K \subseteq M - \{y\}\). If for each pair of distinct points \(x\) and \(y\) of
\(M\), \(M\) is aposyndetic at \(x\) with respect to \(y\) (either one of the
points with respect to the other), then \(M\) is said to be aposyn-
detic (semi-aposyndetic).

In the next lemma we shall use the facts that every gener-
alized tree is semi-aposyndetic, and that every aposyndetic
generalized tree is a tree ([5], Theorem 3.5 and Corollary 2.1).

Lemma 4. If \(M\) is smooth at \(p\) and for each subcontinuum \(N\)
of \(M\) containing \(p\) the components of \(M - N\) form a null family,
then \(M/\mathcal{D}\) is a tree.

Proof. Applying the hypothesis and the properties of \(\phi\)
discussed in Section 2, it is easy to verify that for each sub-
continuum \(K\) of \(M/\mathcal{D}\) which contains \(D(p)\), the components of
\(M/\mathcal{D} - K\) form a null family. Thus it suffices to assume that
\(M\) is a generalized tree (i.e., \(M = M/\mathcal{D}\)), and prove that \(M\) is
aposyndetic. Let \(x\) and \(y\) be distinct points of \(M\). Since \(M\) is
semi-aposyndetic, we can assume that there is a subcontinuum \(H\)
of \(M\) such that \(y \in \text{int}(H) \subseteq H \subseteq M - \{x\}\). If \(x \leq y\), then \(M\) is
aposyndetic at \(x\) with respect to \(y\) ([5], Corollary 3.6), and
the proof is complete. Otherwise, \(x \notin py \cup H\). Let \(C\) denote
the components of \(M - (py \cup H)\), and let \(C\) denote the member of
\(C\) containing \(x\). If \(x \notin \text{int}(C)\), then there is a net \(x_n\) in
M - (py ∪ H ∪ C) such that lim x_n = x. Let C_n be the corresponding net in C (i.e., x_n ∈ C_n), and assume without loss of generality that C_n converges. Since x_n ∉ C for each n, C_n is not eventually constant. But x ∈ lim C_n and \((\lim C_n) ∩ (py ∪ H) ≠ ∅\), which contradicts the assumption that C is a null family. Consequently, \(x ∈ \text{int}(C) ⊆ \text{cl}(C) ⊆ M - \{y\}\), and M is aposyndetic.

**Lemma 5.** Let M be smooth at p and assume that for each subcontinuum N of M containing p, N is accessible and the components of \(M - N\) form a null family. If N is a subcontinuum of M containing p and C is a component of \(M - N\), then \(N ∩ \text{cl}(C)\) is degenerate.

**Proof.** Suppose that N and C are as in the hypothesis and that \(N ∩ \text{cl}(C)\) is nondegenerate. According to Lemma 4, \(M/\emptyset\) is a tree and thus Lemma 3 applies. Consequently C is open and \(M - C\) is a subcontinuum of M containing p. By hypothesis and Lemma 3(b), there is a unique point \(x ∈ M - C\) which is accessible from each point of C. Observe that \(x ∈ N ∩ \text{cl}(C)\). Let H be a nondegenerate subcontinuum of M such that \(N ∩ H = \{x\}\) and \(H - \{x\} ⊆ C\). Let \(y ∈ N ∩ \text{cl}(C)\) such that \(y ≠ x\), and let \(y_n\) be a net in \(C - H\) converging to y. Arguing as above, we conclude that for each n there is a unique point \(z_n ∈ N ∪ H\) which is accessible from \(y_n\). Since C is continuumwise connected (Lemma 3(a)) and M is hereditarily unicoherent at p it follows easily that \(z_n ∈ H - N\) for each n. Let \(C_n\) denote the component of \(M - (H ∪ N)\) which contains \(y_n\). Passing to a subnet if necessary, assume that \(C_n\) converges to a continuum \(C_0\). Since \(y ∈ C_0\) and \(C_0 ∩ H ≠ ∅\), the net \(C_n\) must be eventually constant. It follows that \(z_n\) is eventually constant, say \(z_n = z_0\) for sufficiently large n. Thus \(z_0 ∈ py_n\) for sufficiently large n; and by smoothness \(z_0 ∈ py ⊆ N\). Thus \(z_0 ∈ N\) and \(z_0 ∈ H - N\) which is a contradiction.
Proof of Theorem 1. (Only if) By the Lemma in [6], M is hereditarily unicoherent at p; and by Lemma 1 and 2, each irreducible subcontinuum of the form px is smooth at p. To show that M is smooth at p it suffices to prove that \( < \) is closed in \( M \times M \) ([5], Theorem 3.1). Let \( (x_n, y_n) \) be a net in \( < \) converging to \( (x, y) \). Let \( r: M \to px \cup py \) be a monotone retraction. By [4], Theorem 4.1, \( r \) preserves order, and hence \( r(x_n) \leq r(y_n) \) for each \( n \). Since px and py are smooth at p, so is \( px \cup py \); and consequently \( x = r(x) \leq r(y) = y \). Thus \( (x, y) \) belongs to \( < \) and M is smooth at p.

Let \( N \) be any subcontinuum of M containing p, and let \( r: M \to N \) be a monotone retraction. If \( x \in M - N \), then \( r^{-1}(r(x)) \cap N = \{r(x)\} \), so \( N \) is accessible.

We next show that \( M/\mathfrak{D} \) is a tree. By [12], Theorem 3, it suffices to show that each arc of the form \( D(p)D(x) \) in \( M/\mathfrak{D} \) is a monotone retract of \( M/\mathfrak{D} \). Let \( r: M \to px \) be a monotone retraction. Since \( r \) preserves order, it is easy to verify that the induced map \( r^*: M/\mathfrak{D} \to D(p)D(x) \) defined by \( r^*(D(y)) = D(r(y)) \) for each \( y \in M \) is a monotone retraction.

Finally, let \( N \) be a subcontinuum of M containing p and let \( C \) denote the components of \( M - N \). Assume that \( C_n \) is a net of elements of \( C \) which is not eventually constant and converges to a subcontinuum C. Since each \( C_n \) is open by Lemma 3, it follows that \( C \subseteq N \). Let \( r: M \to N \) be a monotone retraction. Then, by Lemma 3, \( r(C_n) \) is degenerate for each \( n \). Hence \( C = r(C) = \lim r(C_n) \) is degenerate; i.e., \( C \) forms a null family.

(If) Let \( N \) be an subcontinuum of M which contains p. We must define a monotone retraction \( r: M \to N \). For each \( x \in M - N \), denote by \( C(x) \) the component of \( M - N \) containing \( x \). Define \( r: M \to N \) to be the unique retraction such that for each \( x \in M - N \), \( \{r(x)\} = N \cap \text{cl}(c(x)) \). Note that \( r \) is a well-defined function by Lemma 5. Since point inverses of \( r \) are clearly connected, it
remains only to show that \( r \) is continuous. If not, there exists an open set \( U \) in the relative topology on \( N \) such that \( r^{-1}(U) \) is not open in \( M \). Let \( z \in r^{-1}(U) - \text{int}(r^{-1}(U)) \). Applying Lemma 4 and Lemma 3, it follows that \( C(x) \) is open for each \( x \), and thus \( z \in U \subseteq N \). Consequently, there is a net \( z_n \) in \( M - N \) such that \( z_n \notin r^{-1}(U) \) for each \( n \), \( \lim z_n = z \), and \( (N \cap \text{cl}(C(z_n))) \cap U = \emptyset \) for each \( n \). Without loss of generality, assume that the net \( C(z_n) \) converges to a continuum \( C \). The net \( C(z_n) \) is not eventually constant; for otherwise \( z_n \in r^{-1}(z) \subseteq r^{-1}(U) \) for sufficiently large \( n \). But \( C \) contains \( z \) and meets \( N - U \), contradicting the assumption that the components of \( M - N \) form a null family. Thus \( r \) is continuous.

**Corollary 1.** Let \( M \) be a generalized tree which is smooth at \( p \). Then \( M \) is a tree if and only if for each subcontinuum \( N \) of \( M \) containing \( p \), the components of \( M - N \) form a null family.

*Proof.* (Only if) If \( M \) is a tree then each subcontinuum of \( M \) is a monotone retract of \( M \) ([10], Theorem 2.1), and Theorem 1 applies.

(If) By Lemma 4, \( M/\emptyset = M \) is a tree.

**Corollary 2.** Let \( M \) be a continuum which is irreducible about a finite set. Each subcontinuum of \( M \) which contains \( p \) is a monotone retract of \( M \) (i.e., \( M \) satisfies (**)) if and only if

(a) \( M \) is smooth at \( p \), and

(b) each subcontinuum of \( M \) containing \( p \) is accessible.

Furthermore, if (**), holds, then \( M/\emptyset \) is a finite tree.

*Proof.* If \( N \) is a subcontinuum of \( M \) which contains \( p \), then the components of \( M - N \) form a finite, hence null, family. Now apply Theorem 1.

4. Examples

Corollary 2 shows that the "null family" condition in
Theorem 1 is superfluous for continua irreducible about finitely many points. The following example shows that this condition cannot be omitted in general, even if $M/\mathcal{G}$ is known to be a tree.

**Example 1.** Let $M$ be the plane continuum defined by:

$$M = \{(x,y): y = 1 + \sin \frac{1}{x} \text{ for } -1 < x < 0\}$$

$$\cup \{(x,y): x = 0 \text{ and } 0 \leq y \leq 2\}$$

$$\cup \left(\bigcup_{n=0}^{\infty} \{(x,y): y = nx \text{ for } 0 \leq y \leq 2\}\right).$$

Note that $M$ is the union of a "simple harmonic fan" and a "sin $1/x$ curve." The "sin $1/x$ curve" is not a monotone retract of $M_i$ and $M/\mathcal{G}$ is a locally connected fan (i.e., a dendrite with only one ramification point).

The next example shows that the "accessibility" condition in Theorem 1 cannot be omitted even for irreducible continua.

**Example 2.** Let $M$ be the plane continuum defined by:

$$M = \{(x,y): y = \sin \frac{1}{x} \text{ for } -1 < x < 0 \text{ and } 0 < x \leq 1\}$$

$$\cup \{(x,y): x = 0 \text{ and } -1 \leq y \leq 1\}.$$

Note that $M$ is the union of two "sin $1/x$ curves" with a common limit segment. Neither of the "sin $1/x$ curves" is a monotone retract of $M$. Thus $M$ does not satisfy (**).

5. **Concluding Remarks**

Consider the following weak version of condition (**).

(***) Each subcontinuum of $M$ which is irreducible between a fixed point $p$ and some other point is a monotone retract of $M$.

If $M$ is a dendroid, then (*** ) is equivalent to (**) by [12], Theorem 3.

**Question.** Are conditions (**) and (*** ) equivalent for an arbitrary continuum $M$?
We remark that it is possible to modify the proof of Theorem 1 to obtain an affirmative answer to this question in the special case when M is hereditarily unicoherent at the point p. Thus it suffices to determine whether a continuum M satisfying (***)) must be hereditarily unicoherent at p.

References

Salem College
Winston-Salem, North Carolina 27108