INVERSE PRESERVATION OF SMALL INDUCTIVE DIMENSION

by

Peter J. Nyikos
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The following result has long been known to Russians and is considered elementary, but the proof does not seem to have appeared in print:

Theorem 1. Let $X$ be a Hausdorff space and let $f: X \to Y$ be a perfect light map. If $Y$ is regular, then $\text{ind } X < \text{ind } Y$.

(A continuous function $f$ is perfect if it is closed and $f^{-1}(y)$ is compact for all $y \in Y$. It is light if $f^{-1}(y)$ is totally disconnected for all $y \in Y$.)

The proof makes use of the following trivial lemma:

Lemma 2. Let $G_1$ and $G_2$ be disjoint open subsets of a space $X$ and let $K$ be a set whose closure is contained in $G_1 \cup G_2$. Then $\text{Bd}(K \cap G_1) = \text{Bd } K \cap G_1$. In particular, if $K$ is clopen, so is $K \cap G_1$.

Proof of Theorem 1. Let $x$ be a point of $X$ and let $F = f^{-1}(f(x))$. Let $U$ be an open neighborhood of $x$. By zero-dimensionality of $F$, there exist disjoint closed sets $F_1$ and $F_2$ such that $x \in F_1 \subseteq U$, $F_1 \cup F_2 = F$. Let $V_1$ and $V_2$ be disjoint open subsets of $X$ containing $F_1$ and $F_2$ respectively. Let $G_1 = V_1 \cap U$, $G_2 = V_2$.

Let $V = G_1 \cup G_2$. Because $f$ is a closed map, $[f(V^C)]^C$ is an open set containing $f(x)$ whose inverse image is contained in $V$.

The rest of the proof goes by induction. Suppose $\text{ind } Y = 0$. Then there exists a clopen set $K$ containing $f(x)$ and contained in $[f(V^C)]^C$. The inverse image of $K$ is a clopen set contained
in $V$; hence by the lemma, $f^{-1}(K) \cap G_1$ is clopen, and we have $x \in f^{-1}(K) \cap G_1 \subseteq G_1 \subseteq U$.

Suppose the theorem has been proven for $\text{ind } Y < n$, and let $\text{ind } Y = n+1$. By regularity of $Y$, there exists a neighborhood $A$ of $f(x)$ whose closure is contained in $[f(V^c)]^c$ and whose boundary is of $\text{ind } \leq n$. Since $\text{Bd } f^{-1}(A) \subseteq f^{-1}(\text{Bd } A)$ by continuity it follows that $\text{Bd } f^{-1}(A)$ has small inductive dimension $\leq n$ by the induction hypothesis. By the lemma, $\text{Bd } f^{-1}(A) \cap G_1 = \text{Bd } (f^{-1}(A) \cap G_1)$, so that $f^{-1}(A) \cap G_1$ is a neighborhood of $x$ contained in $G_1$ (hence in $U$) whose boundary has small inductive dimension $\leq n$, as was to be shown.

The only place in the above proof where "perfect" was used was in getting disjoint closed (and relative open) subsets of $f^{-1}(y)$ into disjoint open subsets of $X$. This can be done in a number of alternative ways. For example (we take "regular" and "normal" to include "Hausdorff"):

**Theorem 2.** Let $X$ be a regular space and let $f: X \to Y$ be a closed map such that $f^{-1}(y)$ is Lindelöf (or locally compact) and zero-dimensional for all $y \in Y$. If $Y$ is regular, then $\text{ind } X \leq \text{ind } Y$.

**Theorem 3.** Let $X$ be a normal space and let $f: X \to Y$ be a closed map such that $f^{-1}(y)$ is zero-dimensional for all $y \in Y$. Then $\text{ind } X \leq \text{ind } Y$.

More generally, we have:

**Theorem 4.** Let $X$ be a topological space and let $f: X \to Y$ be a closed map such that $f^{-1}(y)$ is $C^*$-embedded and zero-dimensional for all $y \in Y$. If $Y$ is regular, then $\text{ind } X \leq \text{ind } Y$.

The following examples show the necessity of "Hausdorff" in Theorem 1 and "normal" in Theorem 3.
Example 5. Let $X$ be the space consisting of a sequence of closed and isolated points $x_n$ which converge to two distinct closed points, $x$ and $z$. Let $Y$ be the space obtained by identifying $x$ and $z$, and let $f$ be the resulting map. (Clearly, $Y$ is homeomorphic to $\omega+1$.) Then $f$ is a perfect light map, and $\text{ind } Y = 0$, but $\text{ind } X = 1$.

Example 6. Let $Z$ be a version of $\Psi$ [2, Exercise 5I] which is zero-dimensional but not strongly zero-dimensional [3]. Let $g: Z \rightarrow [0,1]$ be a continuous function such that $g^{-1}(0)$ and $g^{-1}(1)$ are not contained in disjoint clopen sets. Let $X$ be the space which is gotten by identifying $g^{-1}(1)$ to a single point and letting the neighborhoods of this point have a base consisting of the sets $g^{-1}(1-c,1]$. Let the rest of $X$ be given the relative topology as a subspace of $Z$. Then $X$ is Tychonoff, and $\text{ind } X = 1$.

Let $f: X \rightarrow Y$ be the map resulting from identifying all nonisolated points of $X$ to a single point, $Y$ the resulting space (which is homeomorphic to $\omega+1$). Then $f$ is closed, and $f^{-1}(y)$ is closed and zero-dimensional for all $y \in Y$. But $\text{ind } Y = 0$.

An interesting consequence of Theorem 1 is that the inverse preservation of a class of zero-dimensional spaces under perfect light maps with Hausdorff domain, is equivalent to its inverse preservation under perfect maps with zero-dimensional Hausdorff domain.

Definition 7. Let $\mathcal{A}$ be a category of topological spaces and let $\mathcal{B}$ be a full and replete subcategory of $\mathcal{A}$. Then $\mathcal{B}$ is [lightly] left-fitting in $\mathcal{A}$ if whenever $f: X \rightarrow Y$ is a perfect [light] map with $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, then $X \in \mathcal{B}$.

Theorem 8. Let $\mathcal{B}$ be a category of zero-dimensional Hausdorff spaces. The following are equivalent.

1) $\mathcal{B}$ is lightly left-fitting in the category of Hausdorff
spaces.

(2) \( B \) is left-fitting in the category of zero-dimensional Hausdorff spaces.

(3) \( B \) is closed hereditary, and every product of a space in \( B \) with a zero-dimensional compact Hausdorff space is in \( B \).

Proof. That (1) is equivalent to (2) is immediate from Theorem 1. It is clear that (2) implies (3). To prove that (3) implies (2), one adapts the argument in [1], substituting "zero-dimensional" for "Tychonoff" and \( \xi X \) for \( B X \).

Example 9. The category of N-compact spaces is lightly left-fitting in the category of Hausdorff spaces. (A space is N-compact if it can be embedded as a closed subspace in a product of countable discrete spaces.) This follows from Theorem 8, since (3) is clearly satisfied.

Problem 10. Let \( X \) be a Hausdorff space and let \( f: X \to Y \) be a perfect map such that \( \text{ind} f^{-1}(y) \leq n \) for all \( y \in Y \). Is it true that \( \text{ind} X \leq \text{ind} Y + n \)?

This is the natural generalization of Theorem 1, but the proof of Theorem 1 leans so heavily upon the zero-dimensionality of \( f^{-1}(y) \) that there seems little hope of an affirmative answer here, even if we assume \( X \) and \( Y \) to be hereditarily normal.

References


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