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In [R,Z], it is shown that every normal, locally connected, and locally compact Moore space is metrizable. In answer to a question of Wilder, [W], using the continuum hypothesis, an example is given in [Ru,Z] of a perfectly normal hereditarily separable space which is locally homeomorphic to E^2 but is not metrizable. It remains unknown if every perfectly normal locally Euclidian space is collectionwise normal. In this note we prove the following:

Theorem: If X *is a perfectly normal, locally connected and locally compact* T_2*-space and* $\{H_a | a \in A\}$ *is a discrete collection of closed Lindelöf sets in* X, *then there is a collection of mutually exclusive open sets* $\{D_a | a \in A\}$ *such that* $H_a \subset D_a$ *for each* a *in* A.

Proof: Suppose that each H_a is compact. We will first prove our theorem for this special case. Since X is perfectly normal, there is a sequence $\{U_n\}_{n<\omega}$ of open sets in X so that $H = \bigcup\{H_a | a \in A\} = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \overline{U}_n$. For each i, let $\mathcal{U}_{a,n}$ be the collection of components of U_n which intersect H_a and let $U_{a,n} = \bigcup \mathcal{U}_{a,n}$.

For each a in A, there is an integer N(a) so that $U_{a,N(a)} \cap U_{b,N(a)} = \emptyset$ for all b in A - {a}. To see that this is true, let D be a compact neighborhood of H_a so that $D \cap (H - H_a) = \emptyset$. Since the boundary of D is compact, there is an N so that U_N does not intersect the boundary of D. We may let N(a) = N. Now, for each n, let $H_n = \bigcup\{H_a | N(a) < n\}$. Since X is normal, there is a collection of $\{V_n | n \in \omega\}$ of mutually exclusive open sets so that $H_n \subset V_n$. For each a in A, let $D_a = V_{N(a)} \cap U_{a,N(a)}$. Then $\{D_a | a \in A\}$ is a collection of mutually exclusive open sets so that $H_a \subset D_a$ for each a in A, which proves that the theorem is true if each H_a is compact.

Now, suppose that each H_a is Lindelöf. Since X is locally compact, for each a, there is a collection $\{F_{a,n}\}_{n\in\omega}$ of compact sets so that $H_a = \bigcup_{n\in\omega} F_{a,n}$. By the special case of our theorem that we have already established, there is a discrete collection $\{V_{a,n} | a \in A\}$ of open sets so that $F_{a,n} \subset V_{a,n}$ for each a in A and n in ω. Choose the $V_{a,n}$ so that $\overline{V}_{a,n} \cap H_b = \emptyset$ if $b \neq a$. For each a in A, let $D_a = \bigcup_{n\in\omega}(V_{a,n} - \mathrm{cl}(\bigcup_{j\leq n}\{V_{b,j} | b \in A - \{a\}\}))$. Then $\{D_a | a \in A\}$ is a collection of mutually exclusive open sets so that $H_a \subset D_a$ for each a in A.

Corollary 1: [R,Z]. *Every normal locally compact and locally connected Moore space is metrizable.*

Proof: This follows from the fact that every Moore space is subparacompact; and so, with our theorem, we can show that the space is strongly screenable and hence metrizable by Bing's metrization theorem [B].

In much the same way, we obtain the following corollary:

Corollary 2: $[R,Z]_2$. *Every perfectly normal, locally compact and locally connected* θ*-refinable space is paracompact.*

We leave several questions unanswered:

Question 1: Is every perfectly normal, locally Euclidean space collectionwise normal?

Question 2: Is every locally connected and locally peripherally compact normal Moore space metrizable?

Question 3: Is every locally compact and locally connected normal T_2*-space collectionwise normal with respect to compact sets?*

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