IS $\square^\omega (\omega + 1)$ PARACOMPACT?

by

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If $\{X_n : n \in \omega\}$ is a family of spaces, $\square X_n$, called the box product of those spaces, denotes the Cartesian product of the sets with the topology generated by all sets of the form $\Pi G_n$, where $G_n$ need only be open in each factor space $X_n$. If $X_n = X \forall n \in \omega$, we denote $\square X_n$ by $\square^n X$.

Box products have generated considerable interest during the past ten years, as first as "counter-example producing machines," later, as mathematical objects in their own right. Yet, except for a few surprising counter-examples there have been no non-trivial absolute results. As corollaries to more general results, M. E. Rudin and K. Kunen have proved that if the Continuum Hypothesis (CH) is assumed, then $\square^n (\omega_1+1)$ is paracompact; however, in [6,8] they question what occurs when CH is false. Kunen [6] has proved that if Martin's Axiom (MA) is assumed, then $\square X_n$ is paracompact whenever each $X_n$ is compact first countable; however, as stated in [2], the really interesting case occurs when $\square^n (\omega+1)$ when both CH and MA + $\neg$CH fail, as they do in the "random real" models of Solovay [10]. We prove:

**Theorem 1:** If $\square^n (\omega+1)$ is paracompact $\forall \alpha < \omega_1$, then $\square^n (\omega_1+1)$ is paracompact.

**Theorem 2:** If there exists a $\lambda$-scale in $\omega$, then $\square^n (\omega+1)$ is paracompact.

Suppose that for each $n \in \omega X_n$ is a set, then for each

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1"Box Products" is the title of Chapter X of [9] where all the results attributed by this author to others may be found, if not referenced here.
x ∈ \prod_{n \in \omega} X_n,

\overline{x} = \{ y \in \prod_{n \in \omega} X_n : \exists m \in \omega \ \exists n > m \Rightarrow y(n) = x(n) \}

defines an equivalence relation on X and the ensuing quotient set is denoted by \( \nabla X_n \) and called the reduced Frechet product [6]. If \( \nabla X_n \) is given the quotient topology from \( \Box X_n \), then G_δ-sets are open; therefore, \( \nabla X_n \) is paracompact if, and only if, every open cover has a pairwise-disjoint open refinement.

Kunen first observed [6] that when each \( X_n \) is compact, \( \Box X_n \) is paracompact if, and only if, \( \nabla X_n \) is paracompact.

**Proof of Theorem 1:**

We suppose \( \mathcal{F} \) is a basic open covering of \( \nabla^{\omega}(\omega_1+1) \). For each \( \alpha < \omega_1 \) and \( A \subseteq \omega \), define

\[
A(\alpha)(n) = \begin{cases} [\alpha+1, \omega_1] & \text{if } n \in A \\ [0, \alpha] & \text{if } n \notin A, \end{cases}
\]

\( A(\alpha) = \prod_{n \in \omega} A(\alpha)(n) \), and \( \overline{\overline{A(\alpha)}} = \{ \overline{x} : x \in A(\alpha) \} \). The sets \( \overline{\overline{A(\alpha)}} \) are clopen and form a partition of \( \nabla^{\omega}(\omega_1+1) \) since \( \overline{\overline{A(\alpha)}} \neq \overline{\overline{B(\alpha)}} \) iff \((A - B) \cup (B - A) = \) infinite.

We construct for each \( \alpha < \omega_1 \) a collection \( \mathcal{F}(\alpha) \) satisfying

1. \( G \in \mathcal{F}(\alpha) \Rightarrow G \) is clopen and contained in a member of \( \mathcal{F} \),
2. \( \bigcup \mathcal{F}(\alpha) \) is clopen and \( \mathcal{F}(\alpha) \) is a pairwise disjoint collection,
3. \( \beta < \alpha < \omega_1 \Rightarrow \mathcal{F}(\beta) \subseteq \mathcal{F}(\alpha) \),
4. \( \bigcup \{ \mathcal{F}(\alpha) : \alpha < \omega_1 \} \) is a cover of \( \nabla^{\omega}(\omega_1+1) \).

There is a first \( \lambda \in \omega_1 \) such that \( \overline{\omega(\lambda)} \) is contained in an element of \( \mathcal{F} \), let \( \mathcal{F}(0) = \{ \overline{\omega(\lambda)} \} \) and suppose that for \( \alpha < \omega_1 \) we have constructed \( \mathcal{F}(\beta) \ \forall \beta < \alpha \) to satisfy (1), (2), and (3).

If \( \alpha \) is a limit ordinal, then let

\[ \mathcal{F}(\alpha) = \bigcup \{ \mathcal{F}(\beta) : \beta < \alpha \}. \]

If \( \alpha \) is a non-limit ordinal, suppose \( A \subseteq \omega \) and let

\[ \mathcal{T}(A) = \{ \overline{y} \in \overline{\overline{A(\alpha)}} : y^{-1}(\omega_1) = A \}. \]
Since $T(A)$ is homeomorphic to $\nu^\omega(a+1)^2$ we may find a pairwise disjoint basic open covering $S(A)$ of $T(A)$ to satisfy

(i) $\nu \in S(A), n, m \in A \rightarrow \inf \nu(n) = \inf \nu(m)$ is a successor ordinal $> a + 1$.

(ii) $\nu \in S(A) \Rightarrow \exists G \in T \ni \nu \subseteq G$.

By choosing only one representative $A$ for each equivalence class $A(n)$, we let

$S(n) = S(n-1) \cup \{\nu - S(n-1): \nu \in S(A), A \subseteq \omega\}$.

In order to show $S(n)$ satisfies (1), (2), and (3) we need only show $U S(A)$ is closed for each $A \subseteq \omega$. So we suppose $x \in \nu - S(A)$ and $y \in T(A)$ such that $y(n) = x(n)$ if $n \not\in A$ if $n \in A$.

Now choose $\nu \in S(A)$ such that $y \in \nu$ and define

$V_x(n) = \begin{cases} W(n) & \text{if } x(n) \in W(n) \\ [\alpha + 1, \inf W(n)) & \text{if } x(n) \notin W(n). \end{cases}$

From (i) $x \in V_x \subseteq A(n)$; moreover, if $u \in S(A)$ and $u \neq \nu$, then we may assume

$\prod_{n \in A-n} u(n) \cap \prod_{n \in A-n} \nu(n) = \emptyset$. 

Thus, $\emptyset \cap V_x = \emptyset$. Clearly, $A(n) - U S(A)$ is open and our induction is completed.

To see (4) we observe that $x \in V^\omega(\omega_1 + 1) \Rightarrow$ either $x = \omega_1$ or $\exists$ a first $\alpha \exists$

$\alpha > \sup \{x(n): x(n) \neq \omega_1\}$.

In the first case $x \in U S(0)$, and in the second case $x \in U S(\alpha)$. Therefore, our proof is complete.

If $\lambda$ is an ordinal, a $\lambda$-scale in $\omega$ is an order-preserving injection $\nu: \lambda \rightarrow \omega$ given any $x \in \omega \exists \alpha < \lambda$ with $x(n) < \nu(\alpha)(n)$ for all but finitely many $n \in \omega$. It should be clear that there $T(A)$ may actually be a singleton; however, this causes no disturbance.
can be no \( \omega \)-scales in \( \omega \); however, it is a fact, probably due to Hausdorff, that

\[ \text{CH} \rightarrow \exists \text{ an } \omega_1 \text{-scale in } \omega. \]

However, in the random real models for \( \neg \text{CH} \), with the ground model "satisfying" \( \text{CH} \), there is an \( \omega_1 \)-scale in \( \omega \) [4]. Booth's theorem [9, pg. 40] says

\[ \text{MA} \Rightarrow \exists \text{ a } 2^\omega \text{-scale in } \omega. \]

In Cohen's original model for \( \neg \text{CH} \) there is no \( \lambda \)-scale in \( \omega \).

In [4] S. Hechler has shown that given cardinals \( \lambda \) and \( \kappa \) and a model \( M \) of \( \text{ZFC} \) in which

\[ \omega < \text{cf}(\lambda) < \lambda < \min(2^\omega, \text{cf}(\kappa)) \]

then one can "extend" \( M \) to a model \( N \) in which \( \kappa = 2^\omega \) and \( \omega \) has a \( \lambda \)-scale.

van Douwen [1] and Hechler [3] have examined a number of topological cardinal functions which are implied by or are equivalent to the existence of a \( \lambda \)-scale. Kunen [5] proved

(a) \( \exists \lambda \)-scale in \( \omega \) \( \Rightarrow \) \( \lambda \times \square^{\omega}(\omega+1) \) is not normal,

(b) \( \exists 2^\omega \)-scale in \( \omega \) \( \Rightarrow \) \( \lambda \times \square^{\omega}(\omega+1) \) is normal for any ordinal \( \lambda \) such that \( \text{cf}(\lambda) \neq 2^\omega \).

Recall [7] that a space \( Y \) is \( \lambda \)-metrizable for an ordinal \( \lambda \), \( \text{cf}(\lambda) > \omega \), whenever each \( y \in Y \) has a local base \( \{ B(y,\alpha) : \alpha < \lambda \} \) satisfying

(i) \( \beta < \alpha \Rightarrow B(y,\alpha) \subseteq B(y,\beta) \)

(ii) \( y \in B(z,\alpha) \Rightarrow z \in B(y,\alpha) \)

(iii) \( y \in B(z,\alpha) \Rightarrow B(y,\alpha) \subseteq B(z,\alpha). \)

It is well known that \( \lambda \)-metrizable spaces are paracompact.

Our original proof of Theorem 2, presented during this conference, was similar to the proof of Theorem 1 and made use of:

If there is a \( \lambda \)-scale in \( \omega \), then the intersection of less than \( \text{cf}(\lambda) \) open sets of \( \forall^\omega(\omega+1) \) is open.
We give thanks to Brian Scott who has provided us with the "if" part of the Lemma from which our theorem 2 is immediate.

**Proof of Theorem 2:**

**Lemma:** Let $\lambda$ be a regular cardinal. Then $\mathcal{V}^\omega(\omega+1)$ is $\lambda$-metrizable if, and only if, there is a $\lambda$-scale in $\omega^\omega$.

**Proof:** Suppose $\{B_\alpha : \alpha < \lambda\}$ is a well-ordered decreasing local base at $\omega$. It is easy to find $\{G_\alpha : \alpha < \lambda\} \subseteq \{B_\alpha : \alpha < \lambda\}$ and $\{x_\alpha : \alpha < \lambda\} \subseteq \omega^\omega$. such that whenever $\alpha < \beta < \lambda$,

$$G_\beta \subseteq \bigcap_{n \in \omega} [x_\beta(n), \omega] \subseteq G_\alpha, \text{ and } \{G_\alpha : \alpha < \lambda\} \text{ is a local base at } \omega.$$  

If $\Psi(\alpha) = x_\alpha$, then $\Psi : \lambda \to \omega^\omega$ is a $\lambda$-scale in $\omega^\omega$.

Conversely, suppose $\Psi : \lambda \to \omega^\omega$ is a $\lambda$-scale in $\omega^\omega$. For each $x \in \mathcal{V}^\omega(\omega+1)$, let $d(x, x) = \lambda$, and if $\Psi \neq x$, let

$$d(x, y) = \inf\{\alpha < \lambda : \{n \in \omega : \inf(x(n), y(n)) < \Psi(\alpha)(n) \text{ and } x(n) \neq y(n)\} = \omega\}.$$  

We see that $d : \mathcal{V}^\omega(\omega+1) \times \mathcal{V}^\omega(\omega+1) \to \lambda + 1$ satisfies the criterion of [7, Theorem 4.8(B)], and hence $\mathcal{V}^\omega(\omega+1)$ is $\lambda$-metrizable.

The previous lemma establishes that the $\lambda$-metrizability of $\mathcal{V}^\omega(\omega+1)$ is independent of the axioms of ZFC whenever $\text{cf}(\lambda) > \omega$.

In answer to one of the questions we presented at this conference, Eric van Douwen has recently shown \(^3\) that $\mathcal{V}^\omega(\omega+1)$ in the previous lemma may be replaced by $\mathcal{V} X_n$, whenever each $X_n$ is a compact metrizable space. In answer to another of our questions, Judith Roitman has proved:

In a model of set theory which is an iterated CCC extension of length $\lambda$, $\text{cf}(\lambda) > \omega \Rightarrow \mathcal{V} X_n$ is paracompact if each $X_n$ is regular and separable. Furthermore, if $\lambda$ is regular and $\lambda \geq 2^\omega$ in the ground model, then $\mathcal{V} X_n$ is paracompact whenever each $X_n$ is regular and separable.

\(^3\)Presented at the Ohio University Conference on Topology, May 1976.
is compact first countable.

The following questions are outstanding:

1. Is $\square^\omega(\omega+1)$ always paracompact or normal?
2. Is $\square^\omega(\omega+1)$ normal in any model of ZFC?
3. Can there be a normal non-paracompact box product of compact spaces?
4. Is the box product of countably many compact linearly ordered topological spaces paracompact?

References

1. E. K. van Douwen, *Functions from $\omega$ to $\omega$*, this conference.

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