THE STRUCTURE OF SMALL NORMAL $F$-SPACES

by

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1. Introduction

The principal purpose of this paper is to prove the following theorem about the structure of small normal F-spaces, and to derive some corollaries of it. (We call a space X "small" if |C*(X)| = 2\omega; explanations of other terminology and notation appear below.)

1.1 Theorem. Assume the continuum hypothesis. Let X be a normal F-space such that |C*(X)| = 2\omega. Then:

(a) If X is countably compact then X is compact.
(b) If X is locally compact then X is σ-compact.

All hypothesized topological spaces are assumed to be completely regular and Hausdorff. Throughout this paper we shall use the notation and terminology of the Gillman-Jerison text [4] without further comment. We shall however remind the reader of the definition of a few of the concepts that appear below.

A topological space is called an F-space if its cozero-sets are C*-embedded. A space is extremally disconnected if each of its open sets has an open closure. Each extremally disconnected space is an F-space; see 14N.4 of [4]. A space X is called weakly Lindelöf if given an open cover \( \mathcal{U} \) of X, there is a countable subcollection \( \mathcal{U}' \) of \( \mathcal{U} \) such that \( \bigcup \{ U : U \in \mathcal{U}' \} \) is dense in X. The Stone-Čech compactification of X is denoted by \( \beta X \); the Hewitt realcompactification is denoted by \( \upsilon X \). The cardinality of a set S is denoted by |S|. The countable discrete space is denoted by 1.

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N. The set of bounded real-valued continuous functions on a space X is denoted by $C^*(X)$. If we use the continuum hypothesis $2^{\omega_1} = \omega_1$ in the proof of a theorem we indicate this by writing "[CH]" before the statement of the theorem.

The following two theorems will be used in the sequel. The first appears as part of Theorem 4.6 of [2].

1.2 Theorem [CH]. Let $Y$ be a locally compact $\sigma$-compact non-compact space such that $|C^*(Y)| = 2^{\omega_1}$. Then $\beta Y - Y$ contains no proper dense $C^*$-embedded subspace, and an open subspace of $\beta Y - Y$ is $C^*$-embedded in $\beta Y - Y$ iff it is a cozero-set of $\beta Y - Y$.

The following appears as the non-trivial part of Theorem 2.2 of [13].

1.3 Theorem [CH]. Let $K$ be a compact $F$-space such that $|C^*(K)| = 2^{\omega_1}$. If $X$ is a $C^*$-embedded subspace of $K$ then $X$ is weakly Lindelöf.

2. Proof of Theorem 1.1 and Its Corollaries

Proof of Theorem 1.1. As $X$ is an $F$-space, so is $\beta X$ (see 14.25 of [4]). Since $X$ is $C^*$-embedded in $\beta X$, by 1.3 $X$ is weakly Lindelöf.

To prove 1.1(a), suppose $X$ is not compact. Choose $p \in \beta X - X$ and write $\beta X - \{p\}$ as a union of cozero-sets of $\beta X$. As $X$ is weakly Lindelöf there are countably many of these cozero-sets whose union, when intersected with $X$, yields a dense subspace of $X$. Let $V$ denote this union. Then $V$ is a dense cozero-set of $\beta X$ not containing $p$. As $V$ is $C^*$-embedded in the $F$-space $\beta X$, it follows that $\beta X = \beta V$. As $V$ satisfies the hypotheses imposed on $Y$ in 1.2, by 1.2 $\beta X - V$ has no proper dense $C^*$-embedded subset. We now show that $X - V$ is a proper dense $C^*$-embedded subset of $\beta X - V$, thus obtaining a contradiction.
The closed subspace $X - V$ of the normal space $X$ is $C^*$-embedded in $X$, and hence in $\beta X$, and hence in $\beta X - V$. Furthermore $p \in (\beta X - V) - (X - V)$. To show that $X - V$ is dense in $\beta X - V$, let $A$ be an open subset of $\beta X$ meeting $\beta X - V$. Since $V$ is a cozero-set of $\beta X$ it is an $F_\sigma$-set, so $A - V$ is a non-empty $G_\delta$-set of $\beta X$. As $X$ is countably compact, $(A - V) \cap X \neq \emptyset$ (see 8A.4 and 8.8 of [4]); thus $X - V$ is dense in $\beta X - V$. This contradiction shows that $\beta X - X$ could not have been non-empty, so $X$ is compact.

To prove 1.1(b) first note that $X$ is open in $\beta X$ since $X$ is locally compact (see 3.15(d) of [4]). Write $X$ as a union of cozero-sets of $\beta X$; since $X$ is weakly Lindelöf there is a countable subfamily of these cozero-sets whose union $U$ is a dense cozero-set of $\beta X$ and is thus $C^*$-embedded in $\beta X$. Thus $U \subseteq X \subseteq \beta X = \beta U$, and $U$ satisfies the hypotheses imposed on $Y$ in 1.2. Thus by 1.2 any open $C^*$-embedded subspace of $\beta X - U$ is a cozero-set of $\beta X - U$, and hence is $\sigma$-compact (as $\beta X - U$ is compact). But $X - U$ is open in $\beta X - U$ as $X$ is open in $\beta X$, and $X - U$ is $C^*$-embedded in $\beta X - U$ since $X$ is normal and its closed subspace $X - U$ is therefore $C^*$-embedded in $X$. Thus $X$ is the union of two $\sigma$-compact spaces and hence it is $\sigma$-compact.

We next derive some corollaries to Theorem 1.1. There has been some interest in determining whether a product of normal countably compact spaces need to be countably compact; see for example Problem B15 of [9]. Corollary 2.2 gives an affirmative answer for a special case. Recall that a space is $\omega$-bounded if its countable subsets are relatively compact.

2.1 Corollary [CH]. A normal countably compact F-space is $\omega$-bounded. Hence a product of arbitrarily many normal countably compact F-spaces is countably compact.

Proof. Let $S$ be a countable subset of the normal countably
compact F-space $X$. Then $c^\ell_X S$ is separable, normal and countably compact; as it is $C^*$-embedded in $X$, by 14.26 of [4] it is an F-space. Obviously $|C^*(c^\ell_X S)| = 2^\omega$ so by 1.1(a) $c^\ell_X S$ is compact. The remainder of the corollary follows from the fact that products of $\omega$-bounded spaces are $\omega$-bounded, and $\omega$-bounded spaces are countably compact.

We next use 1.1(a) to prove a generalization of 1.1(a).

2.2 Corollary [CH]. Let $X$ be a normal F-space such that $|C^*(X)| = 2^\omega$. Then $\nu X$ is not locally compact at any point of $\nu X - X$.

Proof. Suppose that $p \in \nu X - X$, $V$ is open in $\nu X$, $p \in V$, and $c^\ell_{\nu X} V$ is compact. By 4.1 of [1] $X \cap c^\ell_{\nu X} V$ is pseudocompact. It is also normal so by 3D.2 of [4] it is countably compact. As $X \cap c^\ell_{\nu X} V$ is $C^*$-embedded in $X$, it is an F-space by 14.25 of [4] and $|C^*(X \cap c^\ell_{\nu X} V)| = 2^\omega$. Hence by 1.1 $X \cap c^\ell_{\nu X} V$ is compact. But $c^\ell_{\nu X} V = c^\ell_{\nu X} (X \cap c^\ell_{\nu X} V)$ and $p \in c^\ell_{\nu X} V - X$. From this contradiction the corollary follows.

Recall that the absolute $E(X)$ of a space $X$ is (the unique) extremally disconnected space that can be mapped onto $X$ by a map that is perfect and irreducible (i.e. the map takes proper closed subsets of $E(X)$ to proper closed subsets of $X$). See [10] and [12] for details. The proof of the following well-known "folk lemma" is straightforward and is not included.

2.3 Lemma. If $P$ is countable compactness, or $\omega$-boundedness, or separability, or local compactness, then a space $X$ has property $P$ iff $E(X)$ has $P$.

There has recently been some interest in determining conditions under which $E(X)$ is normal. Hence the following corollary is of interest.
2.4 Corollary [CH]. Assume that $E(X)$ is normal.

(a) If $X$ is countably compact then $X$ is $\omega$-bounded.

(b) If $X$ is separable and locally compact then $X$ is $\sigma$-compact.

(c) If $X$ is separable and $\cup X$ is locally compact then $X$ is $\sigma$-compact.

(d) If $X$ is separable and countably compact then $X$ is compact.

Proof. (a) this follows immediately from 2.1 and 2.3.

(b) This follows from 1.1(b) and 2.3.

(c) As $\cup X$ is locally compact, by [8], page 237, or [12], Theorem 2.10, $E(\cup X) = \cup E(X)$. Hence by 2.3 $\cup E(X)$ is locally compact. By 2.3 $E(X)$ is separable and so $|C^*(E(X))| = 2^\omega$.

Thus by 2.2 it follows that $\cup E(X) = E(X)$. Thus $E(\cup X) = E(X)$ so $\cup X = X$, i.e. $X$ is locally compact. The result now follows from (b).

(d) This follows immediately from (a) or (c).

Conditions on $X$ equivalent to the local compactness of $\cup X$ may be found in Harris [5].

3. Examples and Questions

The following examples are designed to show that most of the hypotheses of Theorem 1.1 and its corollaries are necessary to their proofs.

3.1 Examples. The space of countable ordinals (with the order topology) is a non-compact space satisfying all the hypotheses of 1.1 except that it is not an $F$-space. Under assumption of the continuum hypothesis, the space $\gamma N - \{\omega_1\}$ constructed by Franklin and Rajagalan in [3] is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not an $F$-space.
3.2 Example [CH]. Let $p \in \beta \mathbb{N} - \mathbb{N}$. Then $\beta \mathbb{N} - \{p\}$ is a separable non-compact space satisfying all the hypotheses of 1.1 except that it is not normal.

3.3 Example. Examples abound of normal non-compact $F_{\sigma}$-spaces $X$ such that $|C^*(X)| = 2^\omega$; a non-compact cozero set of $\beta \mathbb{N} - \mathbb{N}$ is such a space.

3.4 Example. Using the set-theoretic hypothesis $\text{\textbullet}$, which is known to be consistent with the continuum hypothesis (see page 32 of [9]), M. Wage has recently constructed a separable normal extremally disconnected space that is not realcompact (see [11]). This shows that the hypotheses on $X$ in 2.2 do not imply that $X$ must be realcompact. It also shows that local compactness cannot be dropped from the hypothesis of 1.1(b), since $\sigma$-compact spaces are realcompact.

3.5 Example. Kunen and Parsons [7] have recently shown that if $\aleph$ is a weakly compact cardinal, and if $E$ denotes the subspace of $\beta \aleph$ (where $\aleph$ is given the discrete topology) consisting of those ultrafilters that contain a set of cardinality less than $\aleph$, then $E$ is a normal, countably compact, non-compact extremally disconnected space. Hence the assumption that $|C^*(X)| = 2^\omega$ cannot be dropped from Theorem 1.1. We do not know whether it can be replaced by some significantly weaker assumption.

We conclude with two open questions.

3.6 Question. Is [CH] necessary to prove 1.1? Does Theorem 1.1 hold without any special set-theoretic assumptions?

3.7 Question. Is there a "real" example of an extremally disconnected locally compact normal space that is not paracompact? Theorem 1.1(b) says that if $X$ is such a space then $|C^*(X)| > 2^\omega$. 
Example 3.5 says that if one assumes the existence of weakly compact cardinals then such spaces exist. Kunen [6] has recently constructed a "real" normal extremally disconnected subspace of $\beta \omega_1$ (where $\omega_1$ has the discrete topology) that is not collection-wise Hausdorff, and thus not paracompact; however, his example is not locally compact. (By "real" we mean that no special set-theoretic hypotheses are used in the construction.)

If $2^\omega = 2^{\omega_1}$ then the discrete space of cardinality $\omega_1$ becomes a counterexample to 1.1(b). Hence the assumption of the continuum hypothesis cannot be dropped from 1.1(b). I do not know if it can be replaced by the assumption that $2^\omega < 2^{\omega_1}$.

Bibliography


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