**TOPOLOGY PROCEEDINGS**
**Volume 1, 1976**
Pages 261–267
**http://topology.auburn.edu/tp/**

# TOPOLOGICAL REDUCED PRODUCTS AND THE GCH

by

Paul Bankston

**Topology Proceedings**

**Web:** http://topology.auburn.edu/tp/
**Mail:** Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
**E-mail:** topolog@auburn.edu
**ISSN:** 0146-4124

# TOPOLOGICAL REDUCED PRODUCTS AND THE GCH

**Paul Bankston** [1]

Since its inception in the late fifties, the theory of reduced products in model theory and algebra has developed into an active field of research with increasingly many participants. In particular the theory of ultraproducts has provided "algebraic" proofs of the compactness theorem of first order logic, the existence of saturated models of certain kinds; as well as a characterization of the notion of elementary equivalence between models. Copious details can be found in [BS] and [CK].

In our paper [B] we attempted to translate the notion of reduced product into the context of general topology and found, not too surprisingly, that here was a vast untapped source of research problems, many of the type already encountered in the theory of box products. In [B] several parallel problems involving box products and "topological" ultraproducts are explored; and it turns out that the ultraproduct theorems are often either easier than their counterparts to prove or can be proved directly in ZFC without recourse to extra set-theoretic axioms.

Topological reduced products are formed as certain quotients of box products where the equivalence relations in question derive from filters on the index set. In this note we present a result about paracompactness in topological ultraproducts (i.e. where the filter is maximal) and show how this result relates to a known theorem about paracompactness in box products (trivially reduced products via the singleton filter). Both of these results relate directly with the Generalized Continuum Hypothesis (GCH).

---

[1] Research completed while the author was a Post-doctoral Fellow at McMaster University, Hamilton, Ontario.

## 1. Preliminaries

Let $\langle X_\alpha: \alpha < \kappa\rangle$ be a $\kappa$-sequence of spaces. An *open box* is a cartesian product $\Pi_\kappa U_\alpha$ where $U_\alpha \subseteq X_\alpha$ is open. The cartesian product of the $X_\alpha$'s, together with all open boxes form a space called the *box product* $\Pi_\kappa X_\alpha$. Now let D be a filter of subsets of $\kappa$. The *topological reduced product via D* of the $X_\alpha$'s is $\Pi_D X_\alpha = \Pi_\kappa X_\alpha/\sim$ where $f \sim g$ iff $\{\alpha: f(\alpha) = g(\alpha)\} \in D$. If $D = \{\kappa\}$ then $\Pi_D X_\alpha = \Pi_\kappa X_\alpha$; and if D is an ultrafilter then $\Pi_D X_\alpha$ is referred to as a *topological ultraproduct*. The natural projection $\Gamma_D: \Pi_\kappa X_\alpha \to \Pi_D X_\alpha$ is clearly an open map, and $\Gamma_D(\Pi_\kappa U_\alpha) = \Pi_D U_\alpha = \{[f]_D: \{\alpha: f(\alpha) \in U_\alpha\} \in D\}$ is called an *open ultrabox*. One obvious remark: If $\mathcal{B}_\alpha$ is a basis for the topology on $X_\alpha$ and if D is any filter then $\Pi_D \mathcal{B}_\alpha = \{\Pi_D U_\alpha: U_\alpha \in \mathcal{B}_\alpha \text{ all } \alpha < \kappa\}$ is a basis for the reduced product topology.

From here on, all filters D on $\kappa$ will be ultrafilters. If $\lambda$ is a cardinal, we say that D is $\lambda$-*regular* if there is a subset $E \subseteq D$ of power $\lambda$ which is point-finite (i.e. $\bigcap E_0 = \emptyset$ for all infinite $E_0 \subseteq E$). D is *regular* if D is $\kappa$-regular. It is well known that $\kappa^+$-regular ultrafilters cannot exist, that regular ultrafilters do exist in abundance, and that $\omega$-regularity is the same as countable incompleteness.

We let $UP_\kappa$ be the following statement: If $\langle X_\alpha: \alpha < \kappa\rangle$ is a $\kappa$-sequence of regular spaces of weight $\leq \exp(\kappa)$ and if D is any regular ultrafilter on $\kappa$ then $\Pi_D X_\alpha$ is paracompact.

*1.1 Theorem.* $UP_\kappa$ *iff* $\exp(\kappa) = \kappa^+$.

Our proof is presented in the next two sections.

## 2. The "If" Direction

We define a space to be $\lambda$-*open* ($\lambda \geq \omega$ a cardinal) if every intersection of $<\lambda$ open sets is open. If X is any space we define $(X)_\lambda$, the $\lambda$-*modification* of X, to be the space with X as point set and whose open sets are unions of $<\lambda$ intersections of open sets of X. Thus X is $\lambda$-open iff $X = (X)_\lambda$. By way of a category-theoretic aside, the operation $(\cdot)_\lambda$ is functorial (where for $f\colon X \to Y$, $(f)_\lambda$ is the same function f) from the category of spaces and continuous maps to its full subcategory of $\lambda$-open spaces. In fact $(\cdot)_\lambda$ is a co-reflection (i.e. right adjoint to inclusion).

A space is $\lambda$-*Lindelöf* if every open cover has a subcover of power $\leq\lambda$. If X has weight $\leq\lambda$ then clearly X is (hereditarily) $\lambda$-Lindelöf.

*2.1 Lemma. Regular, $\lambda$-Lindelöf, $\lambda$-open spaces are paracompact.*

*Proof.* This statement has been independently observed by several people. A simple proof follows: In the case $\lambda = \omega$, the lemma reduces to a well-known result. Assume $\lambda \geq \omega_1$. We first note that regular $\omega_1$-open spaces are zero-dimensional (in the sense of weak inductive dimension). To see this let $x \in X$ with $U_0$ an open neighborhood of x. By regularity there is an open set $U_1$ with $x \in U_1 \subseteq \bar{U}_1 \subseteq U_0$. Repeat the process obtaining a decreasing sequence of neighborhoods $U_0 \supseteq \bar{U}_1 \supseteq U_1 \supseteq \bar{U}_2 \supseteq U_2 \supseteq \cdots$ of x. Then $U = \bigcap_{n<\omega} U_n = \bigcap_{n<\omega} \bar{U}_n$ is both open and closed.

Now suppose $\mathcal{U}$ is an open cover of X. Since X is zero-dimensional and $\lambda$-Lindelöf we can assume $\mathcal{U}$ is well-orderable in type $\lambda$, say $\langle U_\alpha\colon \alpha < \lambda\rangle$ where each $U_\alpha$ is clopen. Use $\lambda$-openness to refine $\mathcal{U}$ to a clopen partition by letting $V_\alpha = U_\alpha - \bigcup_{\xi<\alpha} U_\xi$ for $\alpha < \lambda$.

*2.2 Lemma. Ultraproducts via $\lambda$-regular ultrafilters are $\lambda^+$-open.*

*Proof.* This is the "only if" half of Theorem 4.1 of [B]. Assume D is $\lambda$-regular on $\kappa$ and let $\langle X_\alpha\colon \alpha < \kappa\rangle$ be a $\kappa$-sequence of spaces. To check $\lambda^+$-openness it clearly suffices to restrict attention to families of basic open sets, so let $\langle \Pi_D U_{\alpha,\xi}\colon \xi < \lambda\rangle$

be such a family in $\Pi_D X_\alpha$ and let $[f]_D \in \bigcap_{\xi<\lambda} \Pi_D U_{\alpha,\xi}$. By $\lambda$-regularity there is a (well-ordered) "regularizing set" $E = \langle J_\xi: \xi < \lambda \rangle \subseteq D$. Let $K_\xi = \{\alpha < \kappa: f(\alpha) \in U_{\alpha,\xi}\} \in D$; and for each $\alpha < \kappa$ let $F(\alpha) = \{\xi < \lambda: \alpha \in J_\xi \cap K_\xi\}$. Each $F(\alpha)$ is finite, so the set $V_\alpha = \bigcap_{\xi \in F(\alpha)} U_{\alpha,\xi}$ is open. We show $[f]_D \in \Pi_D V_\alpha \subseteq \bigcap_{\xi<\lambda} \Pi_D U_{\alpha,\xi}$. Indeed $\{\alpha < \kappa: f(\alpha) \in V_\alpha\} = \kappa \in D$. Now suppose $\eta < \lambda$. Then $\{\alpha < \kappa: V_\alpha \subseteq U_{\alpha,\eta}\} \supseteq \{\alpha < \kappa: \eta \in F(\alpha)\} \supseteq J_\eta \cap K_\eta \in D$. This completes the proof.

*2.3 Lemma. Ultraproducts preserve regularity of spaces.*

*Proof.* In fact all reduced products preserve this property. The proof is quite straightforward.

Now to prove the "if" direction, assume $\exp(\kappa) = \kappa^+$, that $\langle X_\alpha: \alpha < \kappa \rangle$ is a $\kappa$-sequence of regular spaces of weight $\leq \exp(\kappa)$, and that $D$ is a regular ultrafilter on $\kappa$. Then $\Pi_D X_\alpha$ is $\kappa^+$-open by 2.2, regular by 2.3, and $\exp(\kappa)$-Lindelöf since its weight is $\leq |\exp(\kappa)^\kappa| = \exp(\kappa)$. Thus by 2.1, $\Pi_D X_\alpha$ is paracompact (even hereditarily ultraparacompact).

## 3. The "Only If" Direction

Let $\langle X_\alpha: \alpha < \kappa \rangle$ be a sequence of spaces. An open box $\Pi_\kappa U_\alpha$ is a *$\lambda$-box* if $|\{\alpha: U_\alpha \neq X_\alpha\}| < \lambda$. The *$\lambda$-box product* $\Pi_\kappa^\lambda X_\alpha$ uses only the $\lambda$-boxes to form a basis for its topology; so $\Pi_\kappa^\omega X_\alpha$ is the usual Tychonov product, while $\Pi_\kappa^{\kappa^+} X_\alpha$ is the full box product.

Now in [Bo] Borges generalizes a technique originated by A. H. Stone ([S]) to show that for $\kappa$ a regular cardinal the $\kappa$-box product of $\kappa^+$ copies of a discrete space of cardinality $\kappa$ is not normal (Stone showed that $\omega^{\omega_1}$ is not normal). Now it is easy to show that if each $X_\alpha$ is discrete of power $\kappa$, $\kappa$ regular, then $\Pi^\kappa_{\kappa^+} X_\alpha \simeq (\kappa^{\kappa^+})_\kappa$, where $\kappa^{\kappa^+}$ denotes the Tychonov product $\Pi^\omega_{\kappa^+} X_\alpha$. Thus we can state Borges' theorem as follows:

*3.1 Lemma (Borges). Let* $\kappa$ *be a regular cardinal. Then* $(\kappa^{\kappa^+})_\kappa$ *isn't normal.*

In [vD], van Douwen uses 3.1 in the case $\kappa = \omega_1$ to prove that $(2^{\omega_2})_{\omega_1}$ fails to be normal. We use a similar technique to prove the following:

*3.2 Lemma. Let* $\lambda = \kappa^+$. *Then* $(2^{\lambda^+})_\lambda$ *is not normal.*

*Proof.* $\lambda$ is a successor cardinal so is regular. By 3.1 $(\lambda^{\lambda^+})_\lambda$ isn't normal. Now $\lambda \subseteq \exp(\kappa)$ which is discrete and precisely $(2^\kappa)_\lambda$ (since $2^\kappa$ has weight $\kappa$). Thus $(\lambda^{\lambda^+})_\lambda \subseteq (((2^\kappa)_\lambda)^{\lambda^+})_\lambda$ as a closed set so that the larger space isn't normal either. But it is quite straightforward to show that this space is just $((2^\kappa)^{\lambda^+})_\lambda$ which in turn is homeomorphic to $(2^{\kappa\times\lambda^+})_\lambda \simeq (2^{\lambda^+})_\lambda$.

Our strategy is to show that if D is regular on $\kappa$ then the ultrapower $\Pi_D(2^{\kappa^{++}})$ isn't normal. Then, when $\exp(\kappa) \neq \kappa^+$, the space $2^{\kappa^{++}}$ will be a counterexample to $UP_\kappa$. Thus we are done once we show the following:

*3.3 Lemma. Let* X *be any compact* $T_2$ *space,* $\kappa$ *a cardinal, and* D *a regular ultrafilter on* $\kappa$. *Then* $(X)_{\kappa^+}$ *embeds as a closed subset of the ultrapower* $\Pi_D(X)$.

*Proof.* We draw upon the techniques of §7, [B]. The *D-diagonal* map $\Delta_D: X \to \Pi_D(X)$ takes $x \in X$ to the D-equivalence class of the constant sequence f where $f(\alpha) = x$ for all $\alpha < \kappa$. $\Delta_D$ is always one-one; and in the case D is regular we have (see 7.2 [B]) that $\Delta_D$ embeds $(X)_{\kappa^+}$ into $\Pi_D(X)$. Now when X is compact $T_2$ we define the *D-limit* map $\lim_D: \Pi_D(X) \to X$ by $[f]_D \mapsto x$ whenever x is the unique point such that $[f]_D \in \Pi_D(U)$ for every open U containing x (see 7.4 [B]). By 7.1 of [B] $\lim_D$ is continuous; and in view of the fact that $\Pi_D(X)$ is $\kappa^+$-open, so also is $(\lim_D)_{\kappa^+}$. $\text{Lim}_D$ is clearly a left inverse for $\Delta_D$; whence $\Delta_D$

embeds $(X)_{\kappa^+}$ as a retract of $\Pi_D(X)$. Since retracts are always closed, we have our lemma and hence the proof of 1.1.

*Remark 1.* In light of the foregoing proof it can be seen that $UP_\kappa$ can be paraphrased in several logically equivalent forms. $UP_\kappa$ (et al): If $\langle X_\alpha : \alpha < \kappa \rangle$ is a $\kappa$-sequence of $\begin{Bmatrix} \text{regular} \\ \text{normal} \\ \text{compact } T_2 \end{Bmatrix}$ spaces of weight $\leq \exp(\kappa)$ then for $\begin{Bmatrix} \text{some} \\ \text{every} \end{Bmatrix}$ regular ultrafilter D on $\kappa$, the ultraproduct $\Pi_D X_\alpha$ is (hereditarily) $\begin{Bmatrix} \text{paracompact} \\ \text{normal} \end{Bmatrix}$.

*Remark 2.* In [K] it is proved that the CH implies the statement $BP_\omega$ where for general $\kappa$, $BP_\kappa$ says that for any $\kappa$-sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of compact $T_2$ spaces of weight $\leq \exp(\kappa)$, the box product $\Pi_\kappa X_\alpha$ is paracompact. In [vD] it is shown that $(2^{\omega_2})_{\omega_1}$ is not normal. Thus if CH fails then $2^{\omega_2}$ is a counterexample to $BP_\omega$. In any event, by throwing in singletons where necessary, we have that $2^{\omega_2}$ is an honest counterexample to $BP_\kappa$ for $\kappa > \omega$; so that the status of $BP_\kappa$ for any $\kappa$ is also settled.

**References**

[B] P. Bankston, *Ultraproducts in topology*, to appear in General Topology and Its Applications.

[Bo] C. J. R. Borges, *On a counterexample of A. H. Stone*, Quart. J. Math., Oxford (2) 20 (1969), 91-95.

[BS] J. Bell and A. Slomson, *Models and Ultraproducts*, North Holland (1969).

[CK] C. C. Chang and H. J. Keisler, *Model Theory*, North Holland (1973).

[K] K. Kunen, *Box products of compact spaces*, preprint.

[S] A. H. Stone, *Paracompactness and product spaces*, Bull. A.M.S.54 (1948), 977-982.

[vD] E. K. van Douwen, *Another nonnormal box product*, U. Pittsburgh Research Report #75-111 (1975).

University of Kansas

Lawrence, Kansas 66045