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by

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EC^+ HOMEOMORPHISMS OF EUCLIDEAN SPACES

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1. Introduction

In [10], [8], and [9], Kerékjártó, Homma and Kinoshita, and Husch have given topological characterizations for the standard contraction (also called *dilation*) $x \to x/2$. In particular, $h: S^n \twoheadrightarrow S^n$, $n \neq 4,5$, with $h(\infty) = \infty$, is topologically equivalent to the above standard contraction iff $\{h^n\}_{n=-\infty}^{\infty}$ is an equicontinuous family at all points x except $x = 0,\infty$.

In this paper, we characterize certain homeomorphisms of E^n, $n \neq 4,5$, whose families of non-negative iterates form a pointwise equicontinuous collection, while the full family is not pointwise equicontinuous. We note that if h' denotes h extended to S^n, and if h' fails to have equicontinuous powers at $\{0,\infty\}$ exactly, then our results coincide with the above quoted theorem.

Our techniques are completely elementary, except for the use of prime ends in Section 6, to study the action of h on a certain subcontinuum.

Part I of this paper studies the problem for the plane, and Part II considers the problem for higher dimensional Euclidean spaces.

The present report outlines the results and techniques, but does not include full proofs. Details will appear elsewhere.

Definitions and Notations. A homeomorphism h of E^n onto itself is called EC^+ *(uniformly* EC^+*)* iff its family $\{h^n\}_{n\geq 0}$ of non-negative iterates forms a pointwise (uniformly) equicontinuous collection. It is called *EC (uniformly EC)* iff the family of *all* its iterates $\{h^n\}_{n\in I}$ has this property.

If h is a map from a set X to itself, $B \subseteq X$ is called *invariant* iff $h(B) \subseteq B$. If $h(B) = B$, B is called *fully invariant*.

A double arrow ($\longrightarrow\!\!\!\!\!\rightarrow$) denotes an onto function. *The orbit of* $x = \{h^n(x) \mid n \text{ an integer}\}$, and is denoted by $0(x)$. $0^+(x) = \{h^n(x) \mid n \geq 0\}$. "Orientation preserving" will be denoted by "o.p." and "orientation reversing" will be denoted by "o.r." *Continuum* means compact, connected set. The set of fixed and periodic points of the homeomorphism h is denoted by A. Thus $A = \{x \mid h^n(x) = x, \text{ for some integer } n \geq 1\}$.

Standing Assumption. Unless otherwise stated, in Part I, h will be an EC^+ homeomorphism of E^2 onto itself, for which there exists a point x_o such that $0^+(x_o)$ is bounded. In Part II, h has the same properties on E^n.

Part I: THE PLANE

2. Equicontinuity, ε-sequential growths, and invariant disks

The main theorem of this section is Theorem 2.4, which states that the plane can be filled up with an increasing sequence of invariant disks $\{D_n\}_{n=1}^{\infty}$ such that

(1) $D_1 \subseteq \text{Int } D_2 \subseteq D_2 \subseteq \text{Int } D_3 \subseteq \cdots$ and

(2) $h(D_n) \subseteq D_n \subseteq \text{Int } h(D_{n+1})$. Note that $h(D_n)$ is not necessarily a subset of Int D_n.

In order to prove this theorem we use an ε-sequential growth process, together with equicontinuity, to obtain the invariant disks. The authors earlier used similar methods to obtain invariant disks in [2] and [3].

2.1. *Theorem.* [3,1]. *A homeomorphism* h, *of* E^2 *onto itself, with a fixed point, is pointwise* EC *iff it is a rotation or reflection.*

2.2. *Theorem. The positive semi-orbits of bounded sets* *are bounded.*

2.3. *Theorem. Suppose* M *is a continuum,* $h(M) \subseteq M$, *and* W *is a bounded simply connected domain containing* M *such that* $M \subseteq h(W) \subseteq W$. *Then there is a disk* D *such that* $M \subseteq \text{Int } h(D) \subseteq D \subseteq W$.

2.4. *Theorem. There is a sequence of disks* $\{D_n\}_{n=1}^{\infty}$ *such that*

(1) D_1 *contains a fixed point of* h

(2) $h(D_n) \subseteq D_n \subseteq \text{Int } h(D_{n+1})$, *for each* n, *and*

(3) $\bigcup_{n=1}^{\infty} D_n = E^2$.

2.5. *Remark.* It follows from 2.4 (1) that h has a fixed point.

3. The nucleus of h

In this section we define the *nucleus* of h, a certain subset of E^2, and we study the nature of this set and its relationship to the set A of fixed and periodic points of h.

Let D be a disk such that $h(D) \subseteq D$. Then $\bigcap_{p=1}^{\infty} h^p(D)$ is called the *nucleus of D under h.* From Section 2, Theorem 2.4, we may assume that $E^2 = \bigcup_{n=1}^{\infty} D_n$, where $h(D_n) \subseteq D_n \subseteq \text{Int } h(D_{n+1})$. Let $M_n = \bigcap_{p=1}^{\infty} h^p(D_n)$ = the nucleus of D_n, and let $M = \bigcup_{n=1}^{\infty} M_n$. Then M is called the *nucleus of E^2 under h,* or the *nucleus of h.*

3.1. *Theorem. The nucleus* M *of* h *is well-defined and fully invariant.*

3.2. *Theorem. If the nucleus* M *is bounded, then it is a locally connected, non-separating subcontinuum of* E^2. *If* M *is unbounded, it is the countable increasing union of locally connected, fully invariant continua, none of which separate* E^2, *and is itself locally connected, connected, and closed in* E^2.

3.3. *Remark and Example.* We note that the nucleus of E^2 under h is neither the smallest nor largest, closed connected fully invariant set. For example, if h is a homeomorphism which is the identity on the unit disk, and elsewhere, a contraction along rays emanating from the origin, toward the unit circle, then any disk of radius r, $0 \le r \le 1$ is fully invariant, as is any ray emanating from the origin, and unions of such sets. However the nucleus of this homeomorphism is the closed unit disk.

3.4. *Theorem.* [7] h *is* EC *on* M.

3.5. *Theorem.* $M = \{x \mid 0(x)$ *is bounded*$\}$ *and* $M \supseteq A$.

3.6. *Theorem.* M *is bounded iff* A *is bounded iff there exists a disk* D *such that* $h(D) \subseteq$ *Int* D.

4. Topological contractions and imbeddings in flows

In this section we first study the behavior of $\tilde{h}$ on E^2/M in case M is bounded, where $\tilde{h}$ denotes the map induced by h on the quotient space.

We next obtain a generalization of a theorem of Foland. Foland [6] has proved that any o.p. contracting homeomorphism of E^2 onto itself can be imbedded in a flow. We obtain a generalization of this theorem to E^n, for $n \neq 4,5$.

4.1. *Theorem. Let* $h: E^2 \twoheadrightarrow E^2$ *be an* o.p. EC^+ *homeomorphism, whose set* A *of fixed and periodic points is bounded and* $\neq \emptyset$. *Then* M *is a locally connected continuum which doesn't separate* E^2, *and the induced map* $\tilde{h}: E^2/M \twoheadrightarrow E^2/M$ *is conjugate to the contraction* $r: x \to x/2$ *on* E^2, *and thus* $\tilde{h}$ *is* o.p. *and imbeddable in a flow.*

If h *is* o.r., *then* h^2 *has the above property.*

4.2. *Corollary. If* h *is* o.p. *on* E^2 *with* A *bounded and*

$\neq \emptyset$, *then* h *is* EC^+ *iff there exists a locally connected non-separating continuum* M *such that*

(1) $h|M$ *is* EC, *and*

(2) $\tilde{h}:E^2/M \twoheadrightarrow E^2/M$ $(\cong E^2)$ *is conjugate to a contraction.*

4.3. *Theorem. Let* $h:E^n \twoheadrightarrow E^n$, $n \neq 4,5$, *be a contracting,* o.p. *homeomorphism. Then* h *is conjugate to a standard contraction, and thus imbeddable in a flow.*

(Note: h is *contracting* means there exists $\alpha < 1$ such that $d(h(x),h(y)) < \alpha d(x,y)$, for all $x,y \in E^n$.)

5. The action of h on the nucleus M

In this section we study the action of h on M. The main results are Theorems 5.1 and 5.2, which answer the following two questions.

Question 1. Can an arbitrary locally connected, non-separating continuum in E^2 be the nucleus of some EC^+ homeomorphism of E^2?

The answer is yes, and is given in Theorem 5.1.

Question 2. What is the action of h on M?

It turns out that $h|M$ is periodic or M is a disk and $h|M$ is a rotation, and this is given by Theorem 5.2. We discuss the idea of the proof of this theorem, below.

Both of these results are proved using prime end theory on S^2. See [4,5,11,12] for definitions and a discussion of prime ends.

5.1. *Theorem. Any locally connected, non-separating continuum* M *in* E^2 *can be the nucleus of some* EC^+ *homeomorphism* h *of* E^2 *onto itself, in such a way that* $h|M$ *is the identity.*

5.2. *Theorem.* $h|M$ *is periodic if* M *is not a disk. Otherwise* M *is a rotation.*

Discussion of Proof. We use prime end theory by approaching M from the exterior. Let B be the closed unit disk in E^2, and let ϕ:Int B ——>> U = S^2 - M be a 1-1 "C-transformation"; that is, ϕ^{-1} takes crosscuts to crosscuts, and the endpoints of such crosscuts are dense on the unit circle. Since M is locally connected, we are able to show that ϕ can be extended to a continuous function $\overline{\phi}:B \to \overline{U}$. We think of h as being defined on S^2, $h(\infty) = \infty$. Let $\Psi = \phi^{-1}h\phi$, and Ψ:Int B ——>> Int B can be extended to a homeomorphism $\tilde{\Psi}$:B ——>> B, by the prime end theory, since h is a homeomorphism of $\overline{U}$ onto itself.

We then play back and forth between $\tilde{\Psi}|$Bd B and $h|M$. The equicontinuity on M forces $\tilde{\Psi}|$Bd B to be periodic, and this in turn forces h|Bd M to be periodic, and this forces $h|M$ to be periodic, if M is not a disk. Otherwise $\tilde{\Psi}|$Bd B is a rotation, and it follows that h|Bd M is a rotation, as is $h|M$.

6. Some examples

6.1. *Example.* Let f be the example of 3.3, and let g be a rotation. Then h = fg is EC^+ but not EC, and the nucleus of h is the unit disk.

6.2. *Example.* M will again be the unit disk B. $h|B$ will be an irrational rotation α. Outside B, fill up the plane with a continuous collection of spirals closing down on Bd B. If $x \in E^2-B$, x is on some spiral. Then h(x) is obtained by moving along that same spiral in a counter-clockwise direction, thru a rotation α.

PART II: HIGHER DIMENSIONAL EUCLIDEAN SPACES

II.1. Introduction

In this part we study the EC^+, but not EC, homeomorphisms of E^n onto itself, and build up a theory nearly parallel to that for E^2. There are, however, some differences, and we must make some additional assumptions. We begin with the following two standing assumptions:

(1) $h:E^n \longrightarrow\!\!> E^n$ is a homeomorphism which is EC^+.

(2) There exists a point x_o in E^n such that $0^+(x_o)$ is bounded.

Below we define the *nucleus M of E^n under h*, and we make a third basic assumption.

(3) M is bounded.

This is necessary since, for higher dimensional spaces, it is *not* true that the nucleus is bounded iff the set A of fixed and periodic points is bounded. An example is given below to show this.

II.2. Invariant continua

We again use ε-sequential growths and equicontinuity to obtain the theorems of this section.

II.2.1. *Theorem. There exists a sequence of invariant locally connected, non-separating continua* Y_i, *and cells* B_{n_i} *of radius* n_i, *such that*

$$Y_1 \subseteq \text{Int } B_{n_1} \subseteq B_{n_1} \subseteq \text{Int } Y_2 \subseteq Y_2 \subseteq \text{Int } B_{n_2} \subseteq B_{n_2} \subseteq \cdots .$$

II.3. The nucleus M of h

Let $\{Y_i\}_{i\geq 1}$ be the sequence of Theorem II.2.1. Let $M_i = \bigcap_{p=1}^{\infty} h^p(Y_i)$ and let $M = \bigcup_{i=1}^{\infty} M_i$. M_i is called the *nucleus of Y_i* and M is the *nucleus of E^n under h* or *nucleus of h*.

As is the case for E_2, it can be shown that M is well-defined, fully invariant, and locally connected, and h is EC on M. Also M contains A.

II.3.1. *Example*. In this example A is bounded but M is unbounded. Let $h:E^3 \longrightarrow\!\!> E^3$ be $c \circ r$, where r is a fixed irrational rotation on each plane parallel to the yz-plane, with each origin fixed, and c is a contraction toward the yz-plane, along lines parallel to the x-axis. In this case, A is the origin, and therefore bounded, while M is the yz-plane, and thus A is bounded, while M is unbounded.

II.3.2. *Theorem. The nucleus* M *is bounded iff there exists a tame cell* D *such that* $h(D) \subseteq \text{Int } D$.

II.4. Topological contractions and flows

II.4.1. *Theorem. Let* $h: E^n \twoheadrightarrow E^n$ *be an* o.p., EC^+ *homeomorphism,* $n \neq 4,5$, *such that for some tame cell* D, $h(D) \subseteq \text{Int } D$. *Then there exists a cellular continuum* M *such that*

(1) $E^n/M \cong E^n$, *and*

(2) $\tilde{h}: E^n/M \twoheadrightarrow E^n/M$ *is conjugate to a standard contraction, and thus imbeddable in a flow.*

If h *is* o.r., *then* h^2 *has the above properties.*

II.4.2. *Theorem. Let* $h: E^n \twoheadrightarrow E^n$, $n \neq 4,5$ *be an* o.p., EC^+ *homeomorphism, whose nucleus* M *is bounded. Then* $\tilde{h}: E^n/M \twoheadrightarrow E^n/M$ $(\cong E^n)$ *is a topological standard contraction.*

II.5. Open question

What is the action of h *on the nuclei, when* $h: E^n \twoheadrightarrow E^n$, $n > 2$?

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