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SPACES WITH σ-MINIMAL BASES

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In [1] C. E. Aull observed that every quasi-developable space has a σ-minimal base and in [2] he asked if each space with a σ-minimal base was also a quasi-developable space. In this note examples are given which show that spaces with σ-minimal bases need not be quasi-developable. A structural condition is given which forces a space with a σ-minimal base to be a quasi-developable space.

Let N, Q, and R denote the natural numbers, the rational numbers and the real numbers respectively. Also let all spaces be regular T_1-spaces.

Definition 1. A collection $\mathcal{C}$ of sets is said to be *minimal* if whenever $\mathcal{D} \subset \mathcal{C}$ then $\cup\mathcal{D} = \cup\{D \in \mathcal{D}\}$ is a proper subset of $\cup\mathcal{C}$.

In [6] minimal collections are called irreducible collections.

In [1] Aull gave the following definition.

Definition 2. A base $\mathcal{B}$ for a topological space X is a σ-*minimal* base if $\mathcal{B} = \cup\{\mathcal{B}_n: n \in N\}$ and each $\mathcal{B}_n$ is a minimal collection.

Definition 3. A topological space X is a *quasi-developable* space if there is a base $\mathcal{B}$ such that $\mathcal{B} = \cup\{\mathcal{B}_n: n \in N\}$ and if $x \in X$ and 0 is an open set containing x, then there is a natural number n such that

$x \in st(x,\mathcal{B}_n) = \cup\{B \in \mathcal{B}_n: x \in B\} \subseteq 0$. If, for each natural number n, $\mathcal{B}_n$ is a cover of X, then X is said to be a developable space (or a Moore Space if X is T_3).

Quasi-developable spaces have been studied extensively in [3], [4], and [5].

The following theorem illustrates some conditions under which spaces with a σ-minimal base are metrizable spaces. Notice that the same conditions force quasi-developable spaces to be metrizable spaces [3].

Theorem 1. (Aull [1]) *Let* X *be a space with a σ-minimal base. If* X *is hereditarily separable or hereditarily Lindelöf or hereditarily* $\aleph_1$*-compact, then* X *is a second countable space.*

Recall that the Sorgenfrey Line is the real line with a topology generated by the collection $\{[a,b): a,b \in R, a < b\}$ where $[a,b) = \{x \in R: a \leq x < b\}$.

Example 1. There is a hereditarily paracompact space X with a σ-minimal base that has a closed subset that is homeomorphic to the Sorgenfrey Line. Thus X is not a quasi-developable space and the property of having a σ-minimal base is not a hereditary property.

Let $X = \{(x,y) \in R^2: y \geq 0\}$ and let the topology for X be generated by a base $\mathcal{B}$ described as follows:

(i) if $(x,y) \in X$ and $y > 0$, then let $\{(x,y)\} \in \mathcal{B}$, and

(ii) if $(x,0) \in X$ and $n \in N$, then let $B(x,n) \in \mathcal{B}$ where $B(x,n) = \{(a,b): 0 < a - x < \frac{1}{n} \text{ and } b(a - x)^{-1} < \frac{1}{n}\} \cup \{(x,0)\}$. Let $C(x,n) = B(x,n) \cup \{(x + \frac{1}{n}, \frac{1}{n^2})\}$. If

$\mathcal{C}_1 = \{\{(x,y)\}: (x,y) \in X, y > 0\}$ and if, for each natural number $n > 1$, $\mathcal{C}_n = \{C(x,n): x \in R\}$, then $\mathcal{C} = \cup\{\mathcal{C}_n: n \in N\}$ is a σ-minimal base for X. This easily follows since $C(x,n)$ is the only member of $\mathcal{C}_n$ that contains $(x + \frac{1}{n}, \frac{1}{n^2})$. Notice that $Y = \{(x,0) \in X: x \in R\}$ with the relative topology is homeomorphic to the Sorgenfrey Line. Since the Sorgenfrey Line is a non-second countable hereditarily separable space, Y does not have a σ-minimal base. Since quasi-developability is a hereditary property [3] and Y is not quasi-developable, X is not a quasi-developable space.

Recall that a perfect (= closed sets are G_δ) quasi-developable space is developable [3]. Since semi-metric spaces are perfect, the next example illustrates how far removed a space with a σ-minimal base is from a quasi-developable space.

Example 2. There is a hereditarily paracompact semi-metric space X with a σ-minimal base that is not a quasi-developable space.

Let $X = R^2$ and let the topology for X be generated by a base B consisting of sets of the following form:

(i) $\{(x,y)\} \in \mathcal{B}$ if $(x,y) \in X$ and $y \neq 0$, and

(ii) $B(x,n) \in \mathcal{B}$ if $(x,0) \in X$ where, for each $n \in N$, $B(x,n) = \{(a,b) \in X: |a - x| < \frac{1}{n}$ and $|b(x - a)^{-1}| < \frac{1}{n}\}$. With this topology X is the well known "bow-tie" space and is known to be a hereditarily paracompact semi-metric space.

If $(x,0) \in X$ and $n \in N$, let $C(x,n) = B(x,n) \cup \{(x + \frac{1}{n}, \frac{1}{n^2})\}$. Let $C_1 = \{\{(x,y)\} \in X: y \neq 0\}$ and if $n \in N$ and $n \geq 2$, let $\mathcal{C}_n = \{C(x,n): x \in R\}$. It easily follows that

$\mathcal{C} = \cup\{\mathcal{C}_n : n \in N\}$ is a σ-minimal base for X. Since X is a perfect, non-developable space it cannot be quasi-developable.

Example 3. There is a hereditarily paracompact space X with a σ-minimal base that does not satisfy the first axiom of countability.

Let X be the space in Example 2. If the compact subset $A = \{(x,0) \in X: 0 \leq x \leq 1\}$ is identified to a point, the resulting quotient space X/A is a hereditarily paracompact semi-stratifiable space that does not have a countable local base at A. To see that X/A has a σ-minimal base it is sufficient to find a collection $\mathcal{C}$ of open subsets of X such that

(i) each member of $\mathcal{C}$ contains A,

(ii) $\mathcal{C}$ is a σ-minimal collection, and

(ii) whenever $\mathcal{O}$ is an open subset of X that contains A, then there is some $C \in \mathcal{C}$ such that $A \subset C \subseteq \mathcal{O}$.

To find such a collection $\mathcal{C}$ let $\mathcal{F}$ be the collection of all finite open covers of A such that if $F \in \mathcal{F}$, then the members of F are basic open sets of the form B(x,n) where $(x,0) \in A$ and n is an arbitrary natural number. It follows that the cardinality of $\mathcal{F}$ is $c = 2^{\omega}$. If $F_\alpha = \{B(x(i),n(i)): i \in A_\alpha\}$, then let $T_\alpha = \{x(i): i \in A_\alpha\}$. For each $F_\alpha \in \mathcal{F}$ choose some real number $x_\alpha \in [0,1]$ such that $x_\alpha \notin T_\alpha$ and, if $\alpha \neq \rho$, then $x_\alpha \neq x_\rho$. Let y_α be a positive real number such that (x_α, y_α) is a boundary point of $\cup F_\alpha \cap \{(x,y) \in R^2: y > 0\}$ considered as a subset of R^2 with the usual Euclidean topology. For each $n \in N$, let H_n denote the collection of all the y_α's such that $\frac{1}{n+1} < y_\alpha \leq \frac{1}{n}$. If $y_\alpha \in H_n$, let $C_\alpha = (\cup F_\alpha \cap \{(x,y) \in R^2: y < \frac{1}{n+1}\}) \cup \{(x_\alpha, y_\alpha)\}$. Let $\mathcal{C}_n = \{C_\alpha : y_\alpha \in H_n\}$

and let $\mathcal{C} = \cup\{\mathcal{C}_n: n \in N\}$. It is clear that $\mathcal{C}$ is the needed collection since if $y_\alpha \in H_n$, then C_α is the only member of $\mathcal{C}_n$ that contains (x_α, y_α). Thus X/A has a σ-minimal base, but is not a first-countable space.

If $\mathcal{B} = \cup\{\mathcal{B}_n: n \in N\}$ is a σ-minimal base for a space X and if A is any subset of X, let $\mathcal{B}|A = \cup\{\mathcal{B}_n|A: n \in N\}$ where $\mathcal{B}_n|A = \{B \cap A: B \in \mathcal{B}_n\}$.

The following theorem gives a structural condition for a space with a σ-minimal base to be a quasi-developable space.

Theorem 2. *Let* $\mathcal{B}$ *be a* σ-*minimal base for a space* X. *If, for each subset* A *of* X, $\mathcal{B}|A$ *is a* σ-*minimal base for the subspace* A *with the relative topology, then* $\mathcal{B}$ *is a* σ-*disjoint base for* X.

Proof. Let n be arbitrary and let B_1 and B_2 be members of $\mathcal{B}_n$. Since $\mathcal{B}_n$ is a minimal collection, B_1 is not contained in B_2 and B_2 is not contained in B_1. If $B_1 \cap B_2 \neq \phi$, let $y \in B_1 \cap B_2$ and let $x \in B_1$ such that $x \notin B_2$. Let $A = \{x,y\}$. Then $B_n|A$ is not minimal since $B_1 \cap A$ contains $B_2 \cap A$. From this contradiction it follows that $B_1 \cap B_2 = \phi$. Thus $\mathcal{B}$ is a σ-disjoint base for X.

It might be conjectured at this point that if each subspace of a space X has a σ-minimal base, then X is a quasi-developable space. The following example shows that this conjecture is false.

Example 4. There is a hereditarily paracompact semi-metric space X which is not a quasi-developable space but every subspace of X has a σ-minimal base.

Let D be subset of R of cardinality $\aleph_1$. Let X = D ×

$\{t: t = 0 \text{ or } t \in \{\frac{1}{n}: n \in N\}\}$ and let the topology for X be generated by a base $\mathcal{B}$ described as follows:

(i) if $(x,y) \in X$ and $y \neq 0$, then $\{(x,y)\} \in \mathcal{B}$,

(ii) if $(x,0) \in X$ and $n \in N$, then $B(x,n) \in \mathcal{B}$ where $B(x,n) = \{(x,0)\} \cup \{(a,b) \in X: a \neq x \text{ and } \max \{|a - x|, |b|\} < \frac{1}{n}\}$.

It is readily seen that X is a hereditarily paracompact semi-metric space. Thus, if X was a quasi-developable space, it would be a paracompact Moore Space and thus metrizable. If X was metrizable it would have a σ-discrete base $\mathcal{U} = \cup\{\mathcal{U}_n | n \in N\}$. For each $n \in N$, there could be at most countably many members of $\mathcal{U}_n$ intersecting $D\times\{0\}$ since $D\times\{0\}$ is a hereditarily Lindelöf space in the relative topology. Thus there would be a countable subcollection of $\mathcal{U}$ which serves as a base for the points of $D\times\{0\}$ in the space X. It is easily seen that no countable subcollection of $\mathcal{U}$ can act as a base for the points of this subset of X. Thus X is not metrizable and, hence, not quasi-developable.

To see that each subspace of X has a σ-minimal base let A be a non-empty subset of X and consider A with the relative topology. Let

$$A_1 = \{(x,y) \in A: y \neq 0\},$$

$$A_2 = \{(x,0) \in A: B(x,n) \cap A_1 \text{ is uncountable for each } n \in N\},$$

$$A(2,n,k) = \{(x,0) \in A_2: B(x,n) \cap A_1 \cap \{(a,b): b = (n + k + 1)^{-1}\} \text{ is uncountable , and}$$

$$A_3 = A - (A_1 \cup A_2)$$

Let $\mathcal{B}_1 = \{\{(x,y)\}: (x,y) \in A_1\}$

For each pair of natural numbers (n,k) let $f_{(n,k)}$ be a one-to-one map from A(2,n,k) into $\{(a,b) \in A_1: b = (n+k+1)^{-1}\}$

such that if $(x,0) \in A(2,n,k)$, then

$$0 < |f_{(n,k)}(x,0) - (x,(n + k + 1)^{-1})| < \frac{1}{n}$$

Let $C(x,n,k) = \{(x,0)\} \cup \{f_{(n,k)}(x,0)\} \cup (B(x,n + k + 2) \cap A)$, if $(x,0) \in A(2,n,k)$. Let $\mathcal{B}(n,k) = \{C(x,n,k): (x,0) \in A(2,n,k)\}$ and let $\mathcal{B}_2 = \cup\{\mathcal{B}(n,k): (n,k) \in N^2\}$.

For each $(x,0) \in A_3$, there is an open set $\mathcal{O}_x$ containing $(x,0)$ such that $\mathcal{O}_x \cap A_1$ is at most a countable set. Since A_3 is a Lindelöf space in the relative topology, the existence of the $\mathcal{O}_x$'s allows the construction of an open set $\mathcal{O}$ such that A_3 is contained in $\mathcal{O}$ and $\mathcal{O} \cap A$, is a countable set. If r and s are rational numbers, $(t,v) \in \mathcal{O} \cap A_1$ and $n \in N$, let $B(r,s,n,t) = \{(a,b) \in A: r < a < s, |b| < \frac{1}{n}\} - \{(t,b) \in A: b > 0\}$. Note that if $r < x < s$ and $(x,0) \in A_3$, then $B(r,s,n,t)$ is an open set (in the subspace A) containing $(x,0)$. It follows that $\mathcal{B}_3 = \{B(r,s,n,t): r,s \in Q, n \in N, (t,v) \in \mathcal{O} \cap A_1$, and $B(r,s,n,t) \subset \mathcal{O}\}$ is a countable collection of sets. It follows that $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$ is a σ-minimal base for the subspace A and, hence, each subspace of X has a σ-minimal base.

It follows from Example 3 that Theorem 2 cannot be simplified to state that if each subspace of X has a σ-minimal base, then X is a quasi-developable space.

In most classes of spaces that generalize metric spaces compactness is enough to force metrizability. For example, a compact quasi-developable space is metrizable. The following example shows that this is not the case with spaces with σ-minimal bases even in the class of LOTS (= linearly ordered topological spaces). This example further distinguishes a space with a σ-minimal base from a quasi-developable space.

Example 5. There is a compact, connected LOTS with a σ-minimal base that is not quasi-developable.

Let X be the unit square with the topology induced by the lexicographic ordering [7]. It is clear that X is a compact, connected LOTS that is not metrizable and, thus, not quasi-developable.

Since the elements of X are ordered pairs (a,b) of real numbers, let open intervals in X be denoted by $](a,b), (c,d)[$ where (a,b) precedes (c,d) in the lexicographic ordering of X.

Let $X = A_1 \cup A_2 \cup A_3 \cup A_4$ where

$$A_1 = \{(x,0) \in X: 0 < x < 1\},$$
$$A_2 = \{(x,1) \in X: 0 < x < 1\},$$
$$A_3 = \{(x,y) \in X: 0 < x < 1, 0 < y < 1\}, \text{ and}$$
$$A_4 = \{(0,0), (0,1), (1,0), (1,1)\}.$$

For each $n \in \{1,2,3,4\}$, a σ-minimal collection $\mathcal{B}_n$ of open subsets of X will be constructed such that $\mathcal{B}_n$ will be a base for the points of A_n.

Let $T = \{(a,b,c,d) \in Q^4: 0 < a < b < c < d < 1\}$. Since T is countable, $T = \{(a_n,b_n,c_n,d_n) \in Q^4: 0 < a_n < b_n < c_n < d_n < 1, n \in N\}$.

For each $i \in N$, let g_i be a one-to-one mapping of $\{x \in R: a_i < x < b_i\}$ onto $\{x \in R: c_i < x < d_i\}$. If $n \in N$, $(x,0) \in A_1$ and $c_i < x < d_i$, then let $B(x,i,n) =](g_i^{-1}(x),\frac{1}{n+1}),(g_i^{-1}(x),\frac{1}{n})[\cup](c_i,1-\frac{1}{n}),(x,\frac{1}{n})[$. If $n \in N$, $(x,1) \in A_2$ and $a_i < x < b_i$, then let $C(x,i,n) =](x,1-\frac{1}{n}),(b_i,\frac{1}{n})[\cup](g_i(x),\frac{1}{n+1}),(g_i(x),\frac{1}{n})[$. Notice that both $B(x,i,n)$ and $C(x,i,n)$ are open sets in X.

Let $\mathcal{B}(i,n) = \{B(x,i,n): \mathcal{C}_i < x < d_i\}$ and let $\mathcal{C}(i,n) = \{C(x,i,n): a_i < x < b_i\}$ and observe that each of these collections is a minimal collection. Let $\mathcal{B}_1 = \mathcal{B}\{\cup(i,n): (i,n) \in N^2\}$ and $\mathcal{B}_2 = \cup\{\mathcal{C}(i,n): (i,n) \in N^2\}$.

Let $\{O_1, O_2, \cdots\}$ be a countable base for the Euclidean topology on $\{x \in R: 0 < x < 1\}$. For each $x \in \{x \in R: 0 < x < 1\}$ let $D(x,n) = \{x\} \times O_n$. Then $\mathcal{D}_n = \{D(x,n): x \in R, 0 < x < 1\}$ is a pairwise disjoint collection of open subsets of X. Let $\mathcal{B}_3 = \cup\{\mathcal{D}_n: n \in N\}$.

Since X is a first-countable space, let $\mathcal{B}_4$ be a countable collection of open sets such that $\mathcal{B}_4$ contains a local base for each of the points of A_4.

Then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ is the desired σ-minimal base for X.

Recall that a perfect Lindelöf space is a hereditarily Lindelöf space. Thus if the space of Example 5 were perfect, it would be a hereditarily Lindelöf space and, hence, metrizable. Since it is not metrizable, it cannot be perfect.

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