SPACES WITH $\sigma$-MINIMAL BASES

by

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In [1] C. E. Aull observed that every quasi-developable space has a $\sigma$-minimal base and in [2] he asked if each space with a $\sigma$-minimal base was also a quasi-developable space. In this note examples are given which show that spaces with $\sigma$-minimal bases need not be quasi-developable. A structural condition is given which forces a space with a $\sigma$-minimal base to be a quasi-developable space.

Let $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ denote the natural numbers, the rational numbers and the real numbers respectively. Also let all spaces be regular $T_1$-spaces.

Definition 1. A collection $\mathcal{C}$ of sets is said to be minimal if whenever $\mathcal{B} \subset \mathcal{C}$ then $\bigcup \mathcal{B} = \bigcup \{D \in \mathcal{B}\}$ is a proper subset of $\bigcup \mathcal{C}$.

In [6] minimal collections are called irreducible collections.

In [1] Aull gave the following definition.

Definition 2. A base $\mathcal{B}$ for a topological space $X$ is a $\sigma$-minimal base if $\mathcal{B} = \bigcup \{\beta_n : n \in \mathbb{N}\}$ and each $\beta_n$ is a minimal collection.

Definition 3. A topological space $X$ is a quasi-developable space if there is a base $\mathcal{B}$ such that $\mathcal{B} = \bigcup \{\beta_n : n \in \mathbb{N}\}$ and if $x \in X$ and $0$ is an open set containing $x$, then there is a natural number $n$ such that...
If \( x \in \text{st}(x, \beta_n) = \bigcup \{ B \in \beta_n : x \in B \} \subseteq 0 \). If, for each natural number \( n \), \( \beta_n \) is a cover of \( X \), then \( X \) is said to be a developable space (or a Moore Space if \( X \) is \( T_3 \)).

Quasi-developable spaces have been studied extensively in [3], [4], and [5].

The following theorem illustrates some conditions under which spaces with a \( \sigma \)-minimal base are metrizable spaces. Notice that the same conditions force quasi-developable spaces to be metrizable spaces [3].

**Theorem 1.** (Aull [1]) Let \( X \) be a space with a \( \sigma \)-minimal base. If \( X \) is hereditarily separable or hereditarily Lindelöf or hereditarily \( \beta \)-compact, then \( X \) is a second countable space.

Recall that the Sorgenfrey Line is the real line with a topology generated by the collection \( \{ [a,b) : a, b \in \mathbb{R}, a < b \} \) where \( [a,b) = \{ x \in \mathbb{R} : a < x < b \} \).

**Example 1.** There is a hereditarily paracompact space \( X \) with a \( \sigma \)-minimal base that has a closed subset that is homeomorphic to the Sorgenfrey Line. Thus \( X \) is not a quasi-developable space and the property of having a \( \sigma \)-minimal base is not a hereditary property.

Let \( X = \{ (x,y) \in \mathbb{R}^2 : y > 0 \} \) and let the topology for \( X \) be generated by a base \( \beta \) described as follows:

(i) if \( (x,y) \in X \) and \( y > 0 \), then let \( \{ (x,y) \} \in \beta \), and

(ii) if \( (x,0) \in X \) and \( n \in \mathbb{N} \), then let \( B(x,n) \in \beta \)

where \( B(x,n) = \{ (a,b) : 0 < a - x < \frac{1}{n} \text{ and } b(a - x)^{-1} < \frac{1}{n} \} \cup \{ (x,0) \} \). Let \( C(x,n) = B(x,n) \cup \{ (x + \frac{1}{n}, \frac{1}{n^2}) \} \). If
$C_1 = \{(x,y)\} \times (x,y) \in X, y > 0$ and if, for each natural number $n > 1$, $C_n = \{C(x,n) \times x \in R\}$, then $C = \bigcup C_n : n \in N$ is a $\sigma$-minimal base for $X$. This easily follows since $C(x,n)$ is the only member of $C_n$ that contains $(x + \frac{1}{n}, \frac{1}{n^2})$. Notice that $Y = \{(x,0) \times x \in X: x \in R\}$ with the relative topology is homeomorphic to the Sorgenfrey Line. Since the Sorgenfrey Line is a non-second countable hereditarily separable space, $Y$ does not have a $\sigma$-minimal base. Since quasi-developability is a hereditary property [3] and $Y$ is not quasi-developable, $X$ is not a quasi-developable space.

Recall that a perfect (= closed sets are $G_\delta$) quasi-developable space is developable [3]. Since semi-metric spaces are perfect, the next example illustrates how far removed a space with a $\sigma$-minimal base is from a quasi-developable space.

**Example 2.** There is a hereditarily paracompact semi-metric space $X$ with a $\sigma$-minimal base that is not a quasi-developable space.

Let $X = R^2$ and let the topology for $X$ be generated by a base $B$ consisting of sets of the following form:

(i) $(x,y) \in B$ if $(x,y) \in X$ and $y \neq 0$, and

(ii) $B(x,n) \in B$ if $(x,0) \in X$ where, for each $n \in N$,

$$B(x,n) = \{(a,b) \in X: |a - x| < \frac{1}{n} \text{ and } |b(x - a)^{-1}| < \frac{1}{n}\}.$$  

With this topology $X$ is the well known "bow-tie" space and is known to be a hereditarily paracompact semi-metric space.

If $(x,0) \in X$ and $n \in N$, let $C(x,n) = B(x,n) \cup \{(x + \frac{1}{n}, \frac{1}{n^2})\}$. Let $C_1 = \{(x,y) \times x \in X: y \neq 0\}$ and if $n \in N$ and $n \geq 2$, let $C_n = \{C(x,n) \times x \in R\}$. It easily follows that
\[ C = \bigcup \{ C_n : n \in \mathbb{N} \} \] is a \( \sigma \)-minimal base for \( X \). Since \( X \) is a perfect, non-developable space it cannot be quasi-developable.

**Example 3.** There is a hereditarily paracompact space \( X \) with a \( \sigma \)-minimal base that does not satisfy the first axiom of countability.

Let \( X \) be the space in Example 2. If the compact subset \( A = \{(x,0) : 0 \leq x \leq 1\} \) is identified to a point, the resulting quotient space \( X/A \) is a hereditarily paracompact semi-stratifiable space that does not have a countable local base at \( A \). To see that \( X/A \) has a \( \sigma \)-minimal base it is sufficient to find a collection \( C \) of open subsets of \( X \) such that

1. each member of \( C \) contains \( A \),
2. \( C \) is a \( \sigma \)-minimal collection, and
3. whenever \( O \) is an open subset of \( X \) that contains \( A \), then there is some \( C \in C \) such that \( A \subseteq C \subseteq O \).

To find such a collection \( C \) let \( J \) be the collection of all finite open covers of \( A \) such that if \( F \in J \), then the members of \( F \) are basic open sets of the form \( B(x,n) \) where \( (x,0) \in A \) and \( n \) is an arbitrary natural number. It follows that the cardinality of \( J \) is \( c = 2^\omega \). If \( F_\alpha = \{ B(x(i),n(i)) : i \in A_\alpha \} \), then let \( T_\alpha = \{ x(i) : i \in A_\alpha \} \). For each \( F_\alpha \in J \) choose some real number \( x_\alpha \in [0,1] \) such that \( x_\alpha \notin T_\alpha \) and, if \( \alpha \neq \rho \), then \( x_\alpha \neq x_\rho \). Let \( y_\alpha \) be a positive real number such that \( (x_\alpha,y_\alpha) \) is a boundary point of \( UF_\alpha \cap \{(x,y) \in \mathbb{R}^2 : y > 0\} \) considered as a subset of \( \mathbb{R}^2 \) with the usual Euclidean topology.

For each \( n \in \mathbb{N} \), let \( H_n \) denote the collection of all the \( y_\alpha \)'s such that \( \frac{1}{n+1} < y_\alpha < \frac{1}{n} \). If \( y_\alpha \in H_n \), let \( C_\alpha = \{ (x_\alpha,y_\alpha) \} \). Let \( \{ C_\alpha : y_\alpha \in H_n \} \)
and let $C = \bigcup \{C_n: n \in \mathbb{N}\}$. It is clear that $C$ is the needed collection since if $y_\alpha \in H_n$, then $C_\alpha$ is the only member of $C_n$ that contains $(x_\alpha, y_\alpha)$. Thus $X/A$ has a $\sigma$-minimal base, but is not a first-countable space.

If $E = \bigcup \{E_n: n \in \mathbb{N}\}$ is a $\sigma$-minimal base for a space $X$ and if $A$ is any subset of $X$, let $\beta|A = \bigcup \{\beta_n|A: n \in \mathbb{N}\}$ where $

E \Delta A = \{B \cap A: B \in \beta_n\}$.

The following theorem gives a structural condition for a space with a $\sigma$-minimal base to be a quasi-developable space.

**Theorem 2.** Let $\beta$ be a $\sigma$-minimal base for a space $X$. If, for each subset $A$ of $X$, $\beta|A$ is a $\sigma$-minimal base for the subspace $A$ with the relative topology, then $\beta$ is a $\sigma$-disjoint base for $X$.

**Proof.** Let $n$ be arbitrary and let $B_1$ and $B_2$ be members of $\beta_n$. Since $\beta_n$ is a minimal collection, $B_1$ is not contained in $B_2$ and $B_2$ is not contained in $B_1$. If $B_1 \cap B_2 \neq \emptyset$, let $y \in B_1 \cap B_2$ and let $x \in B_1$ such that $x \notin B_2$. Let $A = \{x, y\}$. Then $\beta_n|A$ is not minimal since $B_1 \cap A$ contains $B_2 \cap A$. From this contradiction it follows that $B_1 \cap B_2 = \emptyset$. Thus $\beta$ is a $\sigma$-disjoint base for $X$.

It might be conjectured at this point that if each subspace of a space $X$ has a $\sigma$-minimal base, then $X$ is a quasi-developable space. The following example shows that this conjecture is false.

**Example 4.** There is a hereditarily paracompact semi-metric space $X$ which is not a quasi-developable space but every subspace of $X$ has a $\sigma$-minimal base.

Let $D$ be subset of $\mathbb{R}$ of cardinality $\aleph_1$. Let $X = D \times \ldots$
\{t: t = 0 \text{ or } t \in \{\frac{1}{n}: n \in \mathbb{N}\}\} \text{ and let the topology for } X \text{ be generated by a base } \mathcal{B} \text{ described as follows:}

(i) if \((x,y) \in X \text{ and } y \neq 0\), then \(\{(x,y)\} \in \mathcal{B}\),

(ii) if \((x,0) \in X \text{ and } n \in \mathbb{N}\), then \(B(x,n) \in \mathcal{B}\) where \(B(x,n) = \{(x,0)\} \cup \{(a,b) \in X: a \neq x \text{ and } \max \{|a-x|, |b|\} < \frac{1}{n}\}\).

It is readily seen that \(X\) is a hereditarily paracompact semi-metric space. Thus, if \(X\) was a quasi-developable space, it would be a paracompact Moore Space and thus metrizable. If \(X\) was metrizable it would have a \(\sigma\)-discrete base \(U = \bigcup \{U_n | n \in \mathbb{N}\}\). For each \(n \in \mathbb{N}\), there could be at most countably many members of \(U_n\) intersecting \(D \times \{0\}\) since \(D \times \{0\}\) is a hereditarily Lindelöf space in the relative topology. Thus there would be a countable subcollection of \(U\) which serves as a base for the points of \(D \times \{0\}\) in the space \(X\).

It is easily seen that no countable subcollection of \(U\) can act as a base for the points of this subset of \(X\). Thus \(X\) is not metrizable and, hence, not quasi-developable.

To see that each subspace of \(X\) has a \(\sigma\)-minimal base let \(A\) be a non-empty subset of \(X\) and consider \(A\) with the relative topology. Let

\[A_1 = \{(x,y) \in A: y \neq 0\}\],

\[A_2 = \{(x,0) \in A: B(x,n) \cap A_1 \text{ is uncountable for each } n \in \mathbb{N}\}\],

\[A(2,n,k) = \{(x,0) \in A_2: B(x,n) \cap A_1 \cap \{(a,b): b = (n+k+1)^{-1}\} \text{ is uncountable , and}\]

\[A_3 = A - (A_1 \cup A_2)\]

Let \(\mathcal{B}_1 = \{\{(x,y)\}: (x,y) \in A_1\}\)

For each pair of natural numbers \((n,k)\) let \(f(n,k)\) be a one-to-one map from \(A(2,n,k)\) into \(\{(a,b) \in A_1: b = (n+k+1)^{-1}\}\)
such that if \((x,0) \in \mathcal{A}(2,n,k)\), then
\[0 < |f_{(n,k)}(x,0) - (x,(n+k+1)^{-1})| < \frac{1}{n}\]

Let \(C(x,n,k) = \{(x,0)\} \cup \{f_{(n,k)}(x,0)\} \cup (B(x,n+k+2) \cap \mathcal{A})\), if \((x,0) \in \mathcal{A}(2,n,k)\). Let \(\beta(n,k) = \{C(x,n,k): (x,0) \in \mathcal{A}(2,n,k)\}\) and let \(\beta_2 = \bigcup \{\beta(n,k): (n,k) \in \mathbb{N}^2\}\).

For each \((x,0) \in \mathcal{A}_3\), there is an open set \(O_x\) containing \((x,0)\) such that \(O_x \cap \mathcal{A}_1\) is at most a countable set. Since \(\mathcal{A}_3\) is a Lindelöf space in the relative topology, the existence of the \(O_x\)'s allows the construction of an open set \(O\) such that \(\mathcal{A}_3\) is contained in \(O\) and \(O \cap \mathcal{A}\), is a countable set. If \(r\) and \(s\) are rational numbers, \((t,v) \in O \cap \mathcal{A}_1\) and \(n \in \mathbb{N}\), let
\[B(r,s,n,t) = \{(a,b) \in \mathcal{A}: r < a < s, |a| < \frac{1}{n}\} - \{(t,b) \in \mathcal{A}: b > 0\}.\]
Note that if \(r < x < s\) and \((x,0) \in \mathcal{A}_3\), then \(B(r,s,n,t)\) is an open set (in the subspace \(\mathcal{A}\)) containing \((x,0)\). It follows that \(\beta_3 = \{B(r,s,n,t): r,s \in \mathbb{Q}, n \in \mathbb{N}, (t,v) \in O \cap \mathcal{A}_1\}, and B(r,s,n,t) \subset O\}\) is a countable collection of sets. It follows that \(\beta = \beta_1 \cup \beta_2 \cup \beta_3\) is a \(\sigma\)-minimal base for the subspace \(\mathcal{A}\) and, hence, each subspace of \(X\) has a \(\sigma\)-minimal base.

It follows from Example 3 that Theorem 2 cannot be simplified to state that if each subspace of \(X\) has a \(\sigma\)-minimal base, then \(X\) is a quasi-developable space.

In most classes of spaces that generalize metric spaces compactness is enough to force metrizability. For example, a compact quasi-developable space is metrizable. The following example shows that this is not the case with spaces with \(\sigma\)-minimal bases even in the class of LOTS (= linearly ordered topological spaces). This example further distinguishes a space with a \(\sigma\)-minimal base from a quasi-developable space.
Example 5. There is a compact, connected LOTS with a $\sigma$-minimal base that is not quasi-developable.

Let $X$ be the unit square with the topology induced by the lexicographic ordering [7]. It is clear that $X$ is a compact, connected LOTS that is not metrizable and, thus, not quasi-developable.

Since the elements of $X$ are ordered pairs $(a,b)$ of real numbers, let open intervals in $X$ be denoted by $] (a, b), (c, d) [$ where $(a, b)$ precedes $(c, d)$ in the lexicographic ordering of $X$.

Let $X = A_1 \cup A_2 \cup A_3 \cup A_4$ where

$A_1 = \{(x, 0) \in X: 0 < x < 1\}$,

$A_2 = \{(x, 1) \in X: 0 < x < 1\}$,

$A_3 = \{(x, y) \in X: 0 < x < 1, 0 < y < 1\}$, and

$A_4 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

For each $n \in \{1, 2, 3, 4\}$, a $\sigma$-minimal collection $\beta_n$ of open subsets of $X$ will be constructed such that $\beta_n$ will be a base for the points of $A_n$.

Let $T = \{(a, b, c, d) \in \mathbb{Q}^4: 0 < a < b < c < d < 1\}$. Since $T$ is countable, $T = \{(a_n, b_n, c_n, d_n) \in \mathbb{Q}^4: 0 < a_n < b_n < c_n < d_n < 1, n \in \mathbb{N}\}$.

For each $i \in \mathbb{N}$, let $g_i$ be a one-to-one mapping of $\{x \in \mathbb{R}: a_i < x < b_i\}$ onto $\{x \in \mathbb{R}: c_i < x < d_i\}$. If $n \in \mathbb{N}$, $(x, 0) \in A_1$ and $c_i < x < d_i$, then let $B(x, i, n) = ] (g_i^{-1}(x), \frac{1}{n+1}), (g_i^{-1}(x), \frac{1}{n}) [ \cup (c_i, 1 - \frac{1}{n}), (x, \frac{1}{n}) ]$. If $n \in \mathbb{N}$, $(x, 1) \in A_2$ and $a_i < x < b_i$, then let $C(x, i, n) = ] (x, 1 - \frac{1}{n}), (b_i, \frac{1}{n}) [ \cup (g_i(x), \frac{1}{n+1}), (g_i(x), \frac{1}{n}) ]$. Notice that both $B(x, i, n)$ and $C(x, i, n)$ are open sets in $X$. 
Let $\mathcal{B}(i,n) = \{B(x,i,n): c_i < x < d_i\}$ and let $\mathcal{C}(i,n) = \{C(x,i,n): a_i < x < b_i\}$ and observe that each of these collections is a minimal collection. Let $\mathcal{B}_1 = \mathcal{B}\{U(i,n): (i,n) \in \mathbb{N}^2\}$ and $\mathcal{B}_2 = \cup\{C(i,n): (i,n) \in \mathbb{N}^2\}$.

Let $\{O_1, O_2, \ldots\}$ be a countable base for the Euclidean topology on $\{x \in \mathbb{R}: 0 < x < 1\}$. For each $x \in \{x \in \mathbb{R}: 0 < x < 1\}$ let $D(x,n) = \{x\} \times O_n$. Then $D_n = \{D(x,n): x \in \mathbb{R}, 0 < x < 1\}$ is a pairwise disjoint collection of open subsets of $X$. Let $\mathcal{B}_3 = \cup\{D_n: n \in \mathbb{N}\}$.

Since $X$ is a first-countable space, let $\mathcal{B}_4$ be a countable collection of open sets such that $\mathcal{B}_4$ contains a local base for each of the points of $A_4$.

Then $\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$ is the desired $\sigma$-minimal base for $X$.

Recall that a perfect Lindelöf space is a hereditarily Lindelöf space. Thus if the space of Example 5 were perfect, it would be a hereditarily Lindelöf space and, hence, metrizable. Since it is not metrizable, it cannot be perfect.

References
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