Research Announcement:

A COMPACT NONMETRIZABLE SPACE \( P \) SUCH THAT \( P^2 \) IS COMPLETELY NORMAL

by

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SUCH THAT $P^2$ IS COMPLETELY NORMAL

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In 1948, M. Katětov showed [2] that if $P$ is a compact space such that $P \times P \times P$ is completely normal, then $P$ is metrizable. [Throughout this announcement, the word "space" will always refer to a Hausdorff space.] Katětov then remarked that he did not know whether complete normality of $P \times P$ was enough to give metrizability. I will now give an example of a compact, nonmetrizable space $P$ whose square, assuming Martin's axiom and the negation of the continuum hypothesis [abbreviated MA + \neg CH] is completely normal. I will also indicate why a "naive" example of such a space, if it exists, will be hard to come by.

1. The Example

Let $I = [0,1]$. Let $A \in I$ be of cardinality $\aleph_1$. Replace each $a$ of $A$ by two points, $a^+$ and $a^-$, with the convention that $a^- < a^+$; otherwise we keep the usual order of all of $(I-A) \cup A^+ \cup A^- = P$. Since $P$ is complete in the order topology, it is a compact space, and nonmetrizable because it does not have a countable base: every base must include an open set with $a^+$ as its least element, and one with $a^-$ as its greatest, for all $a \in A$.

2. Preliminary Lemmas

The following oft-used lemma is utilized in the proof:

Lemma 1. Let $H$ and $K$ be subsets of a space $X$. $H$ and
K can be put into disjoint open subsets whenever there exist countable collections of open sets $U_n$ and $V_n$ so that $H \subseteq \bigcup_{n=1}^{\infty} U_n$, $K \subseteq \bigcup_{n=1}^{\infty} V_n$ and $\overline{U}_n \cap K = \emptyset$ and $\overline{V}_n \cap H = \emptyset$ for all $n$.

We also use:

**Lemma 2.** [3] [MA + CH] Let $Y$ be a metric space of cardinality $\aleph_1$. Every subset of $Y$ is an $F_\sigma$.

We also note that $(A^+)^2$ is homeomorphic to a subspace of the Sorgenfrey plane.

**Lemma 3.** [1] [MA + CH] If $X$ is a subspace of the Sorgenfrey plane, of cardinal $\aleph_1$, then $X$ is normal.

**Lemma 4.** The space $P^2 - (A^+ \cup A^-)^2$ is hereditarily Lindelöf.

Now, although $P^2$ is the union of the perfectly normal (under MA + CH) subspaces $P^2 - (A^+ \cup A^-)^2$, $(A^+)^2$, $(A^-)^2$, $A^+ \times A^-$, and $A^- \times A^+$, the whole space is not perfectly normal; for example, the diagonal is not a $G_\delta$. But under MA + CH it is completely normal.

3. **Outline of the Proof**

Let $H$ and $K$ be subsets of $P^2$ such that $H \cap K = \emptyset$, $H \cap \overline{K} = \emptyset$. The objective is to find countable collections $\{U_n\}_n\to\infty$ and $\{V_n\}_n\to\infty$ of open subsets of $P^2$ as in Lemma 1. Using Lemma 4 and regularity of $P^2$, we can get $H - (A^+ \cup A^-)^2$ into a countable union of open sets whose closure misses $K$. By various symmetry arguments, the proof then boils down to showing that there exist countably many open sets whose closures miss $K$ and whose union contains $H \cap (A^+)^2 = H_1$. 
With the help of Lemma 3, we can get $H_1$ into an open subset of $P^2$ whose closure misses $K \cap (A^+)^2$. We will choose all our open sets to be contained in this one. By Lemma 2 [there is a coarser metric topology on $(A^+)^2$] we can let $H_1 = \bigcup_{n=1}^{\infty} F_n$ where each $F_n$ is closed in $(A^+)^2$, and each $x \in F_n$ has a basic first-quadrant neighborhood whose closure misses $K$ and which is a square $1/n$ on a side. For each $F_n$ we cut up $P^2$ into countably many clopen squares $<1/n$ on a side. It is enough to take care of the points of $F_n$ which lie in any one square $S$. Attach the basic $1/n$-neighborhoods to these points, trimming off the parts sticking out of $S$.

The only possible points of $K$ in the closure of the resulting open set lie on a graph which can be thought of as a monotone function. Moreover, points of $K$ can only lie along straight line segments of the graph, of which there are countably many.

Let $E_n$ stand for the one-sided limit points of $F_n \cap S$ on the graph. We use Lemma 2 again to show $F_n \cap S = \bigcup_{n=1}^{\infty} C_{nm}$ where each $C_{nm}$ is closed in the relative Euclidean topology of $E_n \cup (F_n \cap S)$. If we attach the basic $1/n$-neighborhoods to the points of $C_{nm}$ and intersect with $S$, it turns out that the closure of the resulting set misses $K$.

4. What Happens if $2^{\aleph_0} < 2^{\aleph_1}$

The space $P^2$ is not completely normal in any model of set theory where $2^{\aleph_0} < 2^{\aleph_1}$. This is because $P^2$ has the uncountable discrete subspace $\{(a^+,a^-) | a \in A\}$ and is separable, and it is well known that:

Lemma 5. [${\aleph_0 \leq \aleph_1}$] Every discrete subspace of a separable, completely normal space is countable.
This lemma quickly involves us in a famous pair of problems of general topology: whether there exists an S-space (a regular, hereditarily separable space which is not hereditarily Lindelöf) or an L-space (a regular, hereditarily Lindelöf space which is not hereditarily separable). Such spaces have been constructed in some models of set theory; but, in particular, no one has constructed such spaces assuming only $2^{\aleph_0} < 2^{\aleph_1}$.

But from Lemma 5 it is only a short step to:

**Theorem** $[2^{\aleph_0} < 2^{\aleph_1}]$ If $X$ is a compact, nonmetrizable space such that $X^2$ is completely normal, then at least one of the following is true:

1. $X$ is an L-space
2. $X^2$ is an S-space
3. $X^2$ contains both an S-space and an L-space.

If we combine this theorem with the result that MA + CH implies every compact space of countable spread is hereditarily separable, we see that a "naive" example (one whose basic cardinal invariants like density, spread, and hereditary Lindelöf degree do not vary with the model of ZFC used) would have a hereditarily separable square which is not hereditarily Lindelöf (because of not having a $G_\delta$-diagonal). Such a compact space has not yet been constructed in any model of set theory!

Even if we restrict ourselves to $2^{\aleph_0} < 2^{\aleph_1}$, we are out of reasonable candidates: Mary Ellen Rudin showed, after I came up with these results, that the square of a Souslin line is never completely normal.
References


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