Research Announcement:

OPEN RETRACTIONS OF CONTINUA

by

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The origin of this research\(^1\) was the following problem raised by A. R. Stralka in 1972: Is it true that the only open monotone retraction of a dendroid onto an arc is the identity mapping? (Compare \([1]\), p. 266.) A dendroid is an arcwise connected hereditarily unicoherent continuum. By a continuum we mean a connected compact metric space, and a mapping is meant to be a continuous function. Each dendroid is a hereditarily decomposable continuum. A non-trivial open monotone mapping of an irreducible hereditarily unicoherent continuum onto an arc has been constructed by Knaster [3], and Dyer [2] has shown that each such a mapping must possess point-inverses which are indecomposable continua.

Although the solution of Stralka's problem is in the negative (see 2.1 below), we offer a number of positive results under some additional conditions imposed upon continua or mappings (see 1.1-1.6).

A complete version of the present paper will be published elsewhere. The author wishes to thank Professor A. Lelek for continuous encouragement and valuable improvements.

1. Theorems

We use a method developed by Dyer [2] to prove the following result.

1.1. Theorem. Let \(r : X \to Y\) be an open retraction of a

\(^1\)Supported by a graduate fellowship from Wayne State University.
compact metric space \( X \) onto a non-degenerate continuum

\( Y \subset X \) satisfying the conditions:

(i) if \( C \subset X \) is a continuum and \( r(C) \) is non-degenerate,
    then \( C \cap Y \neq \emptyset \),

(ii) if \( y \in Y \) and \( r^{-1}(y) \) is non-degenerate, then there
    exists a continuum \( K \subset r^{-1}(y) \) such that \( y \notin K \) and
    the interior of \( K \) in \( r^{-1}(y) \) is non-empty.

Then \( X = Y \) and \( z \) is the identity mapping.

For monotone retractions of hereditarily unicoherent continua, condition (i) is automatically satisfied. Easy examples show that this condition cannot be omitted in Theorem 1.1. To see that condition (ii) also cannot be omitted, a more complicated construction is needed (see Example 2.1). The next theorem is an analogue of 1.1 for mappings instead of retractions. Here, however, the assumption that the domain space is hereditarily unicoherent has to be included (see Example 2.3). We say that a compact metric space \( X \) is weakly aposyndetic provided, for each point \( x \in X \), there exists a continuum \( K \subset X \) such that \( x \notin K \) and \( \text{Int } K \neq \emptyset \).

1.2. Theorem. Let \( f: X \to Y \) be an open mapping of a hereditarily unicoherent compact metric space \( X \) onto a non-degenerate continuum \( Y \) satisfying the conditions:

(I) if \( C_1, C_2 \subset X \) are continua, \( f(C_1) \) is non-degenerate and \( f(C_1) \subset f(C_2) \), then \( C_1 \cap C_2 \neq \emptyset \),

(II) if \( y \in Y \) and \( f^{-1}(y) \) is non-degenerate, then \( f^{-1}(y) \)
    is weakly aposyndetic.

Then \( f \) is a homeomorphism.
Moreover, condition (I) can be replaced by the assumption that \( f \) is monotone.

The following four theorems are analogues of Theorems 1.1 and 1.2 for quasi-interior and locally confluent mappings [6]. We say that a mapping \( f: X \rightarrow f(X) \) is weakly irreducible provided \( C \neq X \) implies \( f(C) \neq f(X) \) for each continuum \( C \subset X \).

1.3. Theorem. If \( f: X \rightarrow Y \) is a quasi-interior weakly irreducible mapping of a compact metric space \( X \) onto a semi-locally connected continuum \( Y \), then \( f \) is monotone and \( X \) is a continuum. If, in addition, \( f \) is 0-dimensional, then \( f \) is a homeomorphism.

Without assuming that the range continuum is semi-locally connected, Theorem 1.3 is no longer true (see Example 2.4). Also, the condition of \( f \) being quasi-interior and weakly irreducible in 1.3 cannot be replaced by that of being open and 0-dimensional or confluent and weakly irreducible (see Examples 2.5 and 2.6).

1.4. Theorem. If \( f: X \rightarrow Y \) is a locally confluent weakly irreducible mapping of a compact metric space \( X \) onto a locally connected continuum \( Y \), then \( f \) is monotone and \( X \) is a continuum. If, in addition, \( f \) is 0-dimensional, then \( f \) is a homeomorphism.

1.5. Theorem. If \( r: X \rightarrow Y \) is a locally confluent retraction of a compact metric space \( X \) onto a non-degenerate continuum \( Y \subset X \) satisfying condition (i) of Theorem 1.1, then \( r \) is monotone and \( X \) is a continuum. If, in addition,
\( r \) is 0-dimensional, then \( X = Y \) and \( r \) is the identity mapping.

1.6. Theorem. If \( f: X \to Y \) is a locally confluent mapping of a compact metric space \( X \) onto a non-degenerate continuum \( Y \) satisfying condition (I) of Theorem 1.2, then \( f \) is monotone and \( X \) is a continuum. If, in addition, \( f \) is 0-dimensional, then \( f \) is a homeomorphism.

Clearly, conditions (i) and (I) are essential in Theorem 1.5 and 1.6, respectively.

2. Examples

2.1. Example. There exists an open monotone retraction \( r: X \to A \) of a dendroid \( X \subset R^3 \) onto an arc \( A \subset X \) such that \( r \) is not the identity mapping.

During the preparation of this paper,\(^2\) the author has learned that Krasinkiewicz and Minc [4] have independently constructed a similar example solving Stralka's problem. Their construction is, however, different from and less geometrical than ours. We also use Example 2.1 to build another dendroid, as described in 2.2, which provides a solution to a problem raised by A. Lelek (see [5], p. 328).

2.2. Example. There exists an open path-raising mapping \( f: X \to Y \) of a dendroid \( X \subset R^3 \) onto a dendrite \( Y \) such that \( f(C) \not\subset Y \) for each locally connected continuum \( C \subset X \).

2.3. Example. There exists an open monotone mapping \( f: X \to Y \) of a 1-dimensional continuum \( X \) onto an arc \( Y \) such

that both conditions (I) and (II) of Theorem 1.2 are satisfied and $f$ is not a homeomorphism.

2.4. Example. There exists an open weakly irreducible finite-to-one mapping $f$ of the pseudo-arc onto itself such that $f$ is not a homeomorphism.

The existence of a mapping claimed in 2.4 is the result of the existence of a continuous involution on the pseudo-arc.

2.5. Example. There exists a local homeomorphism $f$ of the circle onto itself such that $f$ is not a homeomorphism.

2.6. Example. There exists a confluent weakly irreducible finite-to-one mapping $f: X \to Y$ of a semi-locally connected continuum $X$ onto a semi-locally connected continuum $Y$ such that $f$ is not quasi-interior (hence $f$ is not monotone).

References


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