ON PERFECT SUBPARACOMPACTNESS
AND A METRIZATION THEOREM FOR
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by

J. CHABER AND P. ZENOR
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In this paper, we give a characterization of perfectly
subparacompact spaces and we use this characterization to
prove that perfectly normal subparacompact spaces which are
locally connected and rim-compact are paracompact. In par­
ticular then, every perfectly normal, locally connected, and
rim compact Moore space is metrizable. This generalizes the
main theorem of [R,Z] and answers a question in [A,Z].

Recall that a space X is said to be subparacompact if
for any open cover \( \mathcal{U} \) of X there exists a sequence \( \{V(j)\}_{j \geq 1} \)
of open covers of X such that if \( x \in X \), then there is a \( j \geq 1 \)
so that \( \text{St}(x, V(j)) \subseteq U \) for some \( U \in \mathcal{U} \) (see [Bu]); X is per­
factly subparacompact if X is subparacompact and closed sub­
sets are \( G_\delta \)-sets in X. Also, X is rim-compact if for each
point \( x \) of X and each open set \( V \) containing \( x \), there is an
open set \( W \) with a compact boundary so that \( x \in W \subseteq V \).

We prove

Theorem 1. Every perfectly normal, locally connected,
rim-compact, and subparacompact space is paracompact.

This yields

Corollary. Every perfectly normal, locally connected
and rim-compact Moore space is metrizable.

The proof of Theorem 1 is based on the following
Theorem 2. The following conditions are equivalent for a topological space $X$:

1) $X$ is perfectly subparacompact,

2) for each well-ordered open cover $U$ of $X$ there exists a sequence $\{V(j)\}_{j \geq 1}$ of open covers of $X$ such that if $x \in X$, then there is a $j \geq 1$ such that $St(x, V(j))$ is contained in the first element of $U$ that contains $x$,

3) for each well-ordered open cover $U = \{U_a\}_{a < \gamma}$ of $X$ there exists a sequence $\{E(j)\}_{j \geq 1}$ of closed collections such that $E(j) = \{E_a(j)\}_{a < \gamma}$ is increasing and $\bigcup_{j \geq 1} E_a(j) = \bigcup_{a < \beta} U_a$ for $a < \gamma$,

4) for each well-ordered open cover $U = \{U_a\}_{a < \gamma}$ of $X$ there exists a closed cover $J = \bigcup J(j)$ such that $J(j) = \{F_a(j)\}_{a < \gamma}$ is discrete and $F_a(j) \subseteq U_a \setminus \bigcup_{\beta < a} U_\beta$ for $a < \gamma$.

Theorem 2 was independently proved by H. Junnila.

In the first section, we shall prove Theorem 1 using the implication 1) $\rightarrow$ 4) of Theorem 2. The second section contains a proof of Theorem 2. We end by giving characterizations of perfectly paracompact spaces analogous to those given in Theorem 2.

1. Proof of Theorem 1

Let $X$ be a space satisfying the hypothesis of Theorem 1 and let $U$ be an open cover of $X$. We shall prove that $U$ has a $\sigma$-discrete open refinement.

Since $X$ is rim-compact, we may assume that all elements of $U$ have compact boundaries. Furthermore, we may assume
that \( U = \{ U_\alpha \}_{\alpha < \gamma} \) is well-ordered.

From the condition 4) of Theorem 2 it follows that there exists a closed refinement \( J = \bigcup_{j \geq 1} J(j) \) of \( U \) such that \( J(j) = \{ F_\alpha(j) \}_{\alpha < \gamma} \) is discrete and \( J_\alpha(j) \subseteq U_\alpha \setminus \bigcup_{\beta < \alpha} U_\beta \).

Since \( J \) refines \( U \), it is sufficient to prove that each \( J(j) \) can be expanded to a \( \sigma \)-discrete open family. This follows from the second of the next two lemmas.

**Lemma 1.** If a closed subset \( F \) of a regular rim-compact space \( X \) is contained in an open set \( U \), then there is an open set \( V \) with a compact boundary such that \( F \subseteq V \subseteq \overline{V} \subseteq U \).

**Proof.** Let \( W \) be a finite open cover of the boundary of \( U \) consisting of sets with compact boundaries and such that \( F \cap \overline{W} = \emptyset \). The set \( V = U \setminus \overline{W} \) has the desired properties.

**Lemma 2.** Let \( X \) be a perfectly normal locally connected and rim-compact space. If \( U = \{ U_\alpha \}_{\alpha < \gamma} \) is an open cover of \( X \), \( J = \{ F_\alpha \}_{\alpha < \gamma} \) is a discrete collection of closed subsets of \( X \) and, for \( \alpha < \gamma \)

(i) \( F_\alpha \subseteq U_\alpha \) and the boundary of \( U_\alpha \) is compact,

(ii) \( U_\alpha \cap ( \bigcup_{\beta > \alpha} F_\beta ) = \emptyset \),

then there exists an open collection \( W = \{ W_\alpha \}_{\alpha < \gamma} \) such that \( F_\alpha \subseteq W_\alpha \) and \( W \) is a countable union of discrete collections.

**Proof.** Since \( X \) is perfectly normal, we may choose a sequence \( \{ G_n \}_{n \geq 1} \) of open sets such that

(iii) \( \bigcup J = \bigcap_{n=1}^{\infty} G_n \) and \( G_{n+1} \subseteq G_n \) for \( n \geq 1 \).

Let \( W_{a,n} \) be the union of all components of \( G_n \) intersecting \( F_\alpha \). Since \( X \) is locally connected, each \( W_{a,n} \) is open. Moreover, from the definition of \( W_{a,n} \) it follows that for \( a_1 < a < \gamma \).
(iv) $W_{a_1,n} \cap W_{a,n} = \emptyset$ implies $W_{a_1,n} \cap F_{a} \neq \emptyset$.

We shall show that for each $\alpha < \gamma$ there exists an integer $n$ such that

(v) $W_{a,n} \cap [\bigcup_{\beta > \alpha} F_{\beta}] = \emptyset$.

Assume that this is not the case and take the smallest $\alpha$ such that $W_{a,n} \cap [\bigcup_{\beta > \alpha} F_{\beta}] \neq \emptyset$ for all $n$.

By (i) and Lemma 1 there exists an open set $V$ with a compact boundary such that $F_{\alpha} \subseteq V \subseteq \overline{V} \subseteq U_{\alpha}$. From (ii) it follows that the boundary $C$ of $V$ separates $F_{\alpha}$ from $\bigcup_{\beta > \alpha} F_{\beta}$. Since each $W_{a,n}$ intersects $\bigcup_{\beta > \alpha} F_{\beta}$ and is the union of a collection of connected sets intersecting $F_{\alpha}$, the sets $W_{a,n} \cap C$ are non-void. From the compactness of $C$ and the fact that $G + l \subseteq G$, we have $A = \bigcap_{n>1} \overline{W}_{a,n} \cap C \neq \emptyset$. By (iii) and $W_{a,n} \subseteq G$, $A \subseteq [\bigcup J] \cap C$ but $C \cap [\bigcup_{\beta > \alpha} F_{\beta}] = \emptyset$. Hence there exists an $\alpha_1 < \alpha$ and an $x \in F_{\alpha_1} \cap \bigcap_{n>1} \overline{W}_{a,n}$.

Each $W_{a,n}$ is closed in $G$ and $x \in G$, therefore $x \in \overline{W}_{a,n}$ implies $x \in W_{a,n}$ and (iv) shows that $\alpha_1$ does not satisfy (v) for any $n \geq 1$. This contradicts our choice of $\alpha$.

For each $\alpha < \gamma$, let $n(\alpha)$ be the first integer satisfying (v).

From (iv), the family $\mathcal{W}' = \{W_{a,n} : n(\alpha) = n\}$ is pairwise disjoint. Since each $W_{a,n}$ is a sum of components of $G$, $\mathcal{W}'$ is also discrete in $G$ and, consequently, $\mathcal{W}' = \{G_{n+1} \cap W : W \in \mathcal{W}'\}$ is discrete in $X$. Thus $\mathcal{W} = \bigcup_{n \geq 1} \mathcal{W}_n$ is a $\sigma$-discrete open expansion of $J$.

2. Proof of Theorem 2

It is easy to observe that the conditions 2), 3), and 4) are equivalent and imply 1). We shall prove 1) $\Rightarrow$ 2). In
the proof of Theorem 1 we use 1) $\Rightarrow$ 4). The proof of 2) $\Rightarrow$ 4) is a well known reasoning from [Bi]. H. Junnila gave a direct proof of 1) $\Rightarrow$ 4).

**Proof of 1) $\Rightarrow$ 2).** Let $X$ be a perfectly subparacompact space and let $U = \{U_\alpha\}_{\alpha < \gamma}$ be a well-ordered open cover of $X$. Since $X$ is perfect, we can find, for each $\alpha < \gamma$, a sequence $\{E_\alpha(j)\}_{j \in \mathbb{N}}$ of closed sets such that

1. $\bigcup_{j \in \mathbb{N}} E_\alpha(j) = \bigcup_{\beta < \alpha} U_\beta$

(Note, that in view of 3), we have to modify sets $E_\alpha(j)$ so that, for each $\alpha < \gamma$, $\{E_\alpha(j)\}_{j \in \mathbb{N}}$ forma an increasing collection.)

For each $m \geq 0$ let $N^m$ denote the collection of all sequences of natural numbers of length $m$. If $t \in N^m$ and $i \in N$, then $(t,i) \in N^{m+1}$ denotes an extension of $t$ by $i$. Put $T_u = \bigcup_{n \geq 0} N^{2n}$ and $T_v = \bigcup_{n \geq 0} N^{2n+1}$.

We will define, by induction on $m$, collections $\{U(t)\}_{t \in T_u}$ and $\{V(t)\}_{t \in T_v}$ of open covers of $X$ such that $U(\emptyset) = U(\emptyset \in T_u$ is the only element of $N^0)$ and

1. if $t \in T_u$, then $\{V(t,k)\}_{k \in \mathbb{N}}$ is a sequence of open covers of $X$ such that if $x \in X$, then there is a $k \in N$ such that $\text{St}(x,V(t,k))$ is a subset of some element of $U(t)$,

2. if $t \in T_v$ and $j \in N$, then $U(t,j) = \{U_\alpha(t,j)\}_{\alpha < \gamma}$ where

$$U_\alpha(t,j) = U_\alpha \setminus (E_\alpha(j) \cup \overline{A_\alpha(t)})$$

and

$$A_\alpha(t) = \{x \in X : \text{St}(x,V(t)) \subseteq \bigcup_{\beta < \alpha} U_\beta\}.$$

If $t \in T_v$ and $V(t)$ is an open cover of $X$, then the closure of the set $A_\alpha(t)$ defined in (iii) is contained in $\bigcup_{\beta < \alpha} U_\beta$, for all $\alpha < \gamma$. This, together with (i), shows that
U(t,j) defined in (iii) covers X for all j ∈ N.

If t ∈ T_u, then the collection {V(t,k)}_{k∈N} satisfying (ii) can be found by subparacompactness of X.

Therefore, starting with U(∅) = U, we can define the collections {U(t)}_{t∈T_u} and {V(t)}_{t∈T_v} of open covers of X satisfying (ii) and (iii).

From the fact that T_v is countable, it follows that it is sufficient to prove that if x ∈ X, then there exists a t ∈ T_v such that St(x,V(t)) is a subset of the first element V(t) that contains x.

Suppose that a point x ∈ X does not have the above property and let α_x < γ and t ∈ T_v be such that St(x,V(t)) ⊆ U_α and St(x,V(t)) ⊆ U_α implies α ≥ α_x.

By our assumption x ∈ ∪_{β<α_x} U_β and, by (i), there is a j ∈ N such that x ∈ E_{α_x} (j).

Consider U(t,j). From the definition of the sets U_α(t,j), it follows that x ∉ U_{α_x}(t,j) (by x ∈ E_{α_x} (j)) and x ∉ U_α(t,j) for α > α_x (by St(x,V(t)) ⊆ U_α). Therefore x ∉ ∪_{α>α_x} U_α(t,j).

Since (t,j) ∈ T_u, by (ii), there exists a k and a β < γ such that St(x,V(t,j,k)) ⊆ U_β (t,j). From x ∉ ∪_{α>α_x} U_α(t,j), we have β < α_x, but U_β(t,j) is a subset of U_β and this contradicts our choice of α_x.

3. Theorem 3

The following conditions are equivalent for a T_1 space X:

1) X is perfectly paracompact,

2) for each well-ordered open cover U of X there exists
a sequence \( \{V(j)\}_{j \geq 1} \) of open covers of \( X \) such that if \( x \in X \), then there exist a neighborhood \( O_x \) of \( x \) and \( j > 1 \) such that \( \text{St}(O_x, V(j)) \) is contained in the first element of \( \mathcal{U} \) that contains \( x \) (see [A]).

3) for each well-ordered open cover \( \mathcal{U} = \{U_a\}_{a < \gamma} \) of \( X \) there exists a sequence \( \{\xi(j)\}_{j \geq 1} \) of closed collections such that \( \xi(j) = \{E_a(j)\}_{a < \gamma} \) is increasing and
\[
\bigcup_{j \geq 1} \text{Int} E_a(j) = \bigcup_{j \geq 1} E_a(j) = \bigcup_{\beta < \alpha} U_\beta \text{ for } \alpha < \gamma.
\]

Theorem 3 can be proved in the same way as Theorem 2.

The implication 1) \( \Rightarrow \) 2)(3)) can be also obtained as a corollary to the implication 1) \( \Rightarrow \) 4) from Theorem 2 with the use of collectionwise normality of \( X \).

References


Auburn University
Auburn, AL 36830