ON PERFECT SUBPARACOMPACTNESS AND A METRIZATION THEOREM FOR MOORE SPACES

by

J. CHABER AND P. ZENOR
ON PERFECT SUBPARACOMPACTNESS AND A METRIZATION THEOREM FOR MOORE SPACES

J. Chaber and P. Zenor

In this paper, we give a characterization of perfectly subparacompact spaces and we use this characterization to prove that perfectly normal subparacompact spaces which are locally connected and rim-compact are paracompact. In particular then, every perfectly normal, locally connected, and rim compact Moore space is metrizable. This generalizes the main theorem of [R,Z] and answers a question in [A,Z].

Recall that a space \( X \) is said to be subparacompact if for any open cover \( \mathcal{U} \) of \( X \) there exists a sequence \( \{\mathcal{V}(j)\}_{j \geq 1} \) of open covers of \( X \) such that if \( x \in X \), then there is a \( j \geq 1 \) so that \( \text{St}(x, \mathcal{V}(j)) \subseteq \mathcal{U} \) for some \( \mathcal{U} \in \mathcal{U} \) (see [Bu]); \( X \) is perfectly subparacompact if \( X \) is subparacompact and closed subsets are \( G_\delta \)-sets in \( X \). Also, \( X \) is rim-compact if for each point \( x \) of \( X \) and each open set \( V \) containing \( x \), there is an open set \( W \) with a compact boundary so that \( x \in W \subseteq V \).

We prove

Theorem 1. Every perfectly normal, locally connected, rim-compact, and subparacompact space is paracompact.

This yields

Corollary. Every perfectly normal, locally connected and rim-compact Moore space is metrizable.

The proof of Theorem 1 is based on the following
Theorem 2. The following conditions are equivalent for a topological space $X$:

1) $X$ is perfectly subparacompact,

2) for each well-ordered open cover $\mathcal{U}$ of $X$ there exists a sequence $\{\mathcal{V}(j)\}_{j \geq 1}$ of open covers of $X$ such that if $x \in X$, then there is a $j \geq 1$ such that $\text{St}(x, \mathcal{V}(j))$ is contained in the first element of $\mathcal{U}$ that contains $x$,

3) for each well-ordered open cover $\mathcal{U} = \{U_a\}_{a \prec \gamma}$ of $X$ there exists a sequence $\{\mathcal{E}(j)\}_{j \geq 1}$ of closed collections such that $\mathcal{E}(j) = \{E_a(j)\}_{a \prec \gamma}$ is increasing and $\bigcup_{j \geq 1} E_a(j) = \bigcup_{\beta \prec \alpha} U_\alpha$ for $\alpha < \gamma$,

4) for each well-ordered open cover $\mathcal{U} = \{U_a\}_{a \prec \gamma}$ of $X$ there exists a closed cover $\mathcal{J} = \bigcup \mathcal{J}(j)$ such that $\mathcal{J}(j) = \{F_a(j)\}_{a \prec \gamma}$ is discrete and $F_a(j) \subseteq U_\alpha \setminus \bigcup_{\beta \prec \alpha} U_\beta$ for $\alpha < \gamma$.

Theorem 2 was independently proved by H. Junnila.

In the first section, we shall prove Theorem 1 using the implication 1) $\Rightarrow$ 4) of Theorem 2. The second section contains a proof of Theorem 2. We end by giving characterizations of perfectly paracompact spaces analogous to those given in Theorem 2.

1. Proof of Theorem 1

Let $X$ be a space satisfying the hypothesis of Theorem 1 and let $\mathcal{U}$ be an open cover of $X$. We shall prove that $\mathcal{U}$ has a $\sigma$-discrete open refinement.

Since $X$ is rim-compact, we may assume that all elements of $\mathcal{U}$ have compact boundaries. Furthermore, we may assume
that \( \mathcal{U} = \{ U_{\alpha} \}_{\alpha < \gamma} \) is well-ordered.

From the condition 4) of Theorem 2 it follows that there exists a closed refinement \( \mathcal{J} = \bigcup_{j \geq 1} \mathcal{J}(j) \) of \( \mathcal{U} \) such that
\[
\mathcal{J}(j) = \{ F_{\alpha}(j) \}_{\alpha < \gamma}
\]
is discrete and \( J_{\alpha}(j) \subseteq U_{\alpha} \setminus \bigcup_{\beta < \alpha} U_{\beta} \).

Since \( \mathcal{J} \) refines \( \mathcal{U} \), it is sufficient to prove that each \( \mathcal{J}(j) \) can be expanded to a \( \sigma \)-discrete open family. This follows from the second of the next two lemmas.

**Lemma 1.** If a closed subset \( F \) of a regular rim-compact space \( X \) is contained in an open set \( U \), then there is an open set \( V \) with a compact boundary such that \( F \subseteq V \subseteq \overline{V} \subseteq U \).

**Proof.** Let \( \mathcal{W} \) be a finite open cover of the boundary of \( \mathcal{U} \) consisting of sets with compact boundaries and such that \( F \cap \overline{\mathcal{W}} = \emptyset \). The set \( V = U \setminus \overline{\mathcal{W}} \) has the desired properties.

**Lemma 2.** Let \( X \) be a perfectly normal locally connected and rim-compact space. If \( \mathcal{U} = \{ U_{\alpha} \}_{\alpha < \gamma} \) is an open cover of \( X \), \( \mathcal{J} = \{ F_{\alpha} \}_{\alpha < \gamma} \) is a discrete collection of closed subsets of \( X \) and, for \( \alpha < \gamma \)

(i) \( F_{\alpha} \subseteq U_{\alpha} \) and the boundary of \( U_{\alpha} \) is compact,

(ii) \( U_{\alpha} \cap (\bigcup_{\beta > \alpha} F_{\beta}) = \emptyset \),

then there exists an open collection \( \mathcal{W} = \{ W_{\alpha} \}_{\alpha < \gamma} \) such that \( F_{\alpha} \subseteq W_{\alpha} \) and \( \mathcal{W} \) is a countable union of discrete collections.

**Proof.** Since \( X \) is perfectly normal, we may choose a sequence \( \{ G_{n} \}_{n \geq 1} \) of open sets such that

(iii) \( \bigcup \mathcal{J} = \bigcap_{n=1}^{\infty} G_{n} \) and \( \overline{G}_{n+1} \subseteq G_{n} \) for \( n \geq 1 \).

Let \( W_{\alpha,n} \) be the union of all components of \( G_{n} \) intersecting \( F_{\alpha} \). Since \( X \) is locally connected, each \( W_{\alpha,n} \) is open. Moreover, from the definition of \( W_{\alpha,n} \) it follows that for \( \alpha_{1} < \alpha < \gamma \),
(iv) $W_{a_1,n} \cap W_{a,n} = \emptyset$ implies $W_{a_1,n} \cap F_a \neq \emptyset$.

We shall show that for each $\alpha < \gamma$ there exists an integer $n$ such that

(v) $W_{a,n} \cap [U_{\beta \geq a} F_\beta] = \emptyset$.

Assume that this is not the case and take the smallest $\alpha$ such that $W_{a,n} \cap [U_{\beta \geq a} F_\beta] \neq \emptyset$ for all $n$.

By (i) and Lemma 1 there exists an open set $V$ with a compact boundary such that $F_a \subseteq V \subseteq \overline{V} \subseteq U_\alpha$. From (ii) it follows that the boundary $C$ of $V$ separates $F_a$ from $U_{\beta > a} F_\beta$. Since each $W_{a,n}$ intersects $U_{\beta > a} F_\beta$ and is the union of a collection of connected sets intersecting $F_a$, the sets $W_{a,n} \cap C$ are non-void. From the compactness of $C$ and the fact that $G + l \subseteq G$, we have $A = \cap_{n>1} \overline{W}_{a,n} \cap C \neq \emptyset$. By (iii) and $W_{a,n} \subseteq G$, $A \subseteq [UJ] \cap C$ but $C \cap [U_{\beta > a} F_\beta] = \emptyset$. Hence there exists an $\alpha_1 < \alpha$ and an $x \in F_{\alpha_1} \cap \cap_{n>1} \overline{W}_{a,n}$.

Each $W_{a,n}$ is closed in $G$ and $x \in G$, therefore $x \in \overline{W}_{a,n}$ implies $x \in W_{a,n}$ and (iv) shows that $\alpha_1$ does not satisfy (v) for any $n > 1$. This contradicts our choice of $\alpha$.

For each $\alpha < \gamma$, let $n(\alpha)$ be the first integer satisfying (v).

From (iv), the family $\mathcal{W}' = \{W_{a,n} : n(\alpha) = n\}$ is pairwise disjoint. Since each $W_{a,n}$ is a sum of components of $G$, $\mathcal{W}'$ is also discrete in $G$ and, consequently, $\mathcal{W}' = \{G_{n+1} \cap \mathcal{W} : W \in \mathcal{W}'\}$ is discrete in $X$. Thus $\mathcal{W} = \cup_{n \geq 1} \mathcal{W}_n$ is a $\sigma$-discrete open expansion of $J$.

2. Proof of Theorem 2

It is easy to observe that the conditions 2), 3), and 4) are equivalent and imply 1). We shall prove 1) $\Rightarrow$ 2). In
the proof of Theorem 1 we use 1) → 4). The proof of
2) → 4) is a well known reasoning from [Bi]. H. Junnila
gave a direct proof of 1) → 4).

Proof of 1) → 2). Let $X$ be a perfectly subparacompact
space and let $\mathcal{U} = \{U_\alpha\}_{\alpha<\gamma}$ be a well-ordered open cover of $X$.
Since $X$ is perfect, we can find, for each $\alpha<\gamma$, a sequence
$\{E_\alpha(j)\}_{j\in\mathbb{N}}$ of closed sets such that

(i) $\bigcup_{j\in\mathbb{N}} E_\alpha(j) = \bigcup_{\beta<\alpha} U_\beta$

(Note, that in view of 3), we have to modify sets $E_\alpha(j)$ so
that, for each $\alpha<\gamma$, $\{E_\alpha(j)\}_{j\in\mathbb{N}}$ forma an increasing collec­tion.)

For each $m \geq 0$ let $N^m$ denote the collection of all
sequences of natural numbers of length $m$. If $t \in N^m$ and
$i \in N$, then $(t,i) \in N^{m+1}$ denotes an extension of $t$ by $i$.
Put $T_u = \bigcup_{n \geq 0} N^2n$ and $T_v = \bigcup_{n \geq 0} N^{2n+1}$.

We will define, by induction on $m$, collections $\{U(t)\}_{t\in T_u}$
and $\{V(t)\}_{t\in T_v}$ of open covers of $X$ such that $U(\emptyset) = U(\emptyset \in T_u$
is the only element of $N^0$) and

(ii) if $t \in T_u$, then $\{V(t,k)\}_{k\in N}$ is a sequence of open
covers of $X$ such that if $x \in X$, then there is a $k \in N$ such
that $St(x,V(t,k))$ is a subset of some element of $U(t)$,

(iii) if $t \in T_v$ and $j \in N$, then $U(t,j) = \{U_\alpha(t,j)\}_{\alpha<\gamma}$
where

$U_\alpha(t,j) = U_\alpha \setminus (E_\alpha(j) \cup \overline{A}_\alpha(t))$ and

$A_\alpha(t) = \{x \in X: St(x,V(t)) \subseteq \bigcup_{\beta<\alpha} U_\beta\}$.

If $t \in T_v$ and $V(t)$ is an open cover of $X$, then the
closure of the set $A_\alpha(t)$ defined in (iii) is contained in
$\bigcup_{\beta<\alpha} U_\beta$, for all $\alpha<\gamma$. This, together with (i), shows that
\( U(t,j) \) defined in (iii) covers \( X \) for all \( j \in \mathbb{N} \).

If \( t \in T' \), then the collection \( \{ V(t,k) \}_{k \in \mathbb{N}} \) satisfying (ii) can be found by subparacompactness of \( X \).

Therefore, starting with \( U(\emptyset) = U \), we can define the collections \( \{ U(t) \}_{t \in T_u} \) and \( \{ V(t) \}_{t \in T_v} \) of open covers of \( X \) satisfying (ii) and (iii).

From the fact that \( T_v \) is countable, it follows that it is sufficient to prove that if \( x \in X \), then there exists a \( t \in T_v \) such that \( St(x,V(t)) \) is a subset of the first element \( U \) that contains \( x \).

Suppose that a point \( x \in X \) does not have the above property and let \( \gamma_x < \gamma \) and \( t \in T_v \) be such that \( St(x,V(t)) \subseteq U^{\alpha_x} \) and \( St(x,V(t)) \subseteq U^\alpha \) implies \( \alpha \geq \alpha_x \).

By our assumption \( x \in U^{\beta < \alpha_x} \) and, by (i), there is a \( j \in \mathbb{N} \) such that \( x \in E_{\alpha_x}^j \).

Consider \( U(t,j) \). From the definition of the sets \( U^\alpha(t,j) \), it follows that \( x \notin U^{\alpha_x}(t,j) \) (by \( x \in E_{\alpha_x}^j \)) and \( x \notin U^\alpha(t,j) \) for \( \alpha > \alpha_x \) (by \( St(x,V(t)) \subseteq U^{\alpha_x} \)). Therefore \( x \notin U^{\alpha > \alpha_x}(t,j) \).

Since \( (t,j) \in T_u \), by (ii), there exists a \( k \) and a \( \beta < \gamma \) such that \( St(x,V(t,j,k)) \subseteq U^{\beta}(t,j) \). From \( x \notin U^{\alpha > \alpha_x}(t,j) \), we have \( \beta < \alpha_x \), but \( U^{\beta}(t,j) \) is a subset of \( U^{\beta} \) and this contradicts our choice of \( \alpha_x \).

3. Theorem 3

The following conditions are equivalent for a \( T_1 \) space \( X \):

1) \( X \) is perfectly paracompact,

2) for each well-ordered open cover \( U \) of \( X \) there exists
a sequence \( \{V(j)\}_{j \geq 1} \) of open covers of \( X \) such that if \( x \in X \), then there exist a neighborhood \( O_x \) of \( x \) and a \( j > 1 \) such that \( \text{St}(O_x, V(j)) \) is contained in the first element of \( U \) that contains \( x \) (see [A]).

3) for each well-ordered open cover \( U = \{U_a\}_{a < \gamma} \) of \( X \) there exists a sequence \( \{\xi(j)\}_{j \geq 1} \) of closed collections such that \( \xi(j) = \{E_a(j)\}_{a < \gamma} \) is increasing and

\[
\bigcup_{j \geq 1} \text{Int} E_a(j) = \bigcup_{j \geq 1} E_a(j) = \bigcup_{\beta < a} U_\beta \quad \text{for} \quad a < \gamma.
\]

Theorem 3 can be proved in the same way as Theorem 2. The implication 1) \( \Rightarrow \) 2) (3)) can be also obtained as a corollary to the implication 1) \( \Rightarrow \) 4) from Theorem 2 with the use of collectionwise normality of \( X \).

References


Auburn University

Auburn, AL 36830