A NOTE ON FIXED POINTS IN TREE-LIKE CONTINUA

by

J. B. FUGATE AND L. MOHLER
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Introduction

A continuum is a compact connected metric space; a tree is a continuum which is the union of a finite collection of arcs, and contains no simple closed curve. A continuum M is tree-like provided that for each $\varepsilon > 0$ there is a tree T and a map $f: M \to T$ such that the inverse image of each point of T has diameter less than $\varepsilon$ (such maps are called $\varepsilon$-maps). It follows from [8] that a continuum is tree-like if, and only if, it is the inverse limit of a sequence of trees with bonding maps which are surjections.

In [1] Bing has asked if tree-like continua have the fixed-point property i.e. does each map of a tree-like continuum into itself have a fixed point? Affirmative answers for special cases of this question may be found in [2], [4], [5], and [6]. This work culminates in [7], where Manka shows that a hereditarily decomposable and hereditarily unicoherent continuum has the fixed point property. (A continuum is hereditarily unicoherent provided that the intersection of each pair of subcontinua is a continuum. Tree-like continua are hereditarily unicoherent; if a continuum is hereditarily decomposable, the converse holds [3].)

Principal Theorem

We proceed to prove our main result. We are indebted to L. Wayne Goodwyn, who suggested this approach.
Theorem. If there is a tree-like continuum $M$ and a fixed-point-free map $f: M \to M$, then there is an indecomposable tree-like continuum $X$ and a homeomorphism $h: X \to X$ such that $h$ does not send any proper subcontinuum of $X$ into itself.

Proof. Using the Brouwer Reduction Theorem, one can see that there is a subcontinuum $Y$ of $M$ which is minimal with respect to being mapped into itself. Clearly, $f[Y] = Y$ and, as a subcontinuum of $M$, $Y$ is tree-like. Let $Z$ be the inverse limit of the sequence $(Y_i, f_i)$, where, for each $i$, $Y_i = Y$ and $f_i$ is $f$ restricted to $Y$. Then $Z$ is a continuum. We will show that $Z$ is tree-like by showing that for each $\varepsilon > 0$, there is an $\varepsilon$-map of $Z$ onto a tree. Suppose then that $\varepsilon > 0$ is fixed, and for each $i$, let $Q_i$ be the projection map of $Z$ onto $Y_i$. There is a positive integer $j$ so that $Q_j$ is an $\varepsilon$-map. Using the compactness of $Y_j$, it is easy to see that there is a $\delta > 0$ so that if $A$ is a subset of $Y_j$ of diameter less than $\delta$, then $\text{diam}(Q_j^{-1}(A)) < \varepsilon$. Since $Y_j$ is tree-like, there is a tree $T$ and a $\delta$-map $p: Y_j \to T$. Then $p \circ Q_j$ is an $\varepsilon$-map of $Z$ into $T$, so $Z$ is tree-like.

We now define a homeomorphism $h: Z \to Z$ by $h(z_1, z_2, z_3, \ldots) = (z_2, z_3, \ldots)$. As a function into a product space, $h$ is clearly continuous. Moreover, if $h(z) = h(x)$, then $\begin{align*}
(z_2, z_3, z_4, \ldots) &= (x_2, x_3, x_4, \ldots) \\
\text{so } z_i &= x_i \text{ if } i \geq 2.
\end{align*}$ Then $z_1 = f(z_2) = f(x_2) = x_1$, so $z = x$, and $h$ is one-to-one.

Also, $h$ is fixed-point free, since if $h(x) = x$ then $\begin{align*}
(x_2, x_3, x_4, \ldots) &= (x_1, x_2, x_3, \ldots) \\
\text{and so } x_2 &= x_1 = f(x_2) \text{ and } f \text{ fixes } x_2,
\end{align*}$ which contradicts our assumption about $f$.

We can again apply the Brouwer Reduction Theorem, and obtain a (necessarily tree-like) subcontinuum $X$ of $Z$ so that
h[X] = X and no proper subcontinuum of X is carried into itself by h. To conclude our argument, we will use a technique of Gray to show that X must be indecomposable. Suppose, to the contrary, that there are proper subcontinua $A_0$ and $B_0$ of X such that $X = A_0 \cup B_0$.

For each positive integer $i$, let $A_i = h^{-i}[A_0]$ and $B_i = h^{-i}[B_0]$. The two sequences $A_0, A_1, \ldots$ and $B_0, B_1, \ldots$ have the following properties:

1) For each $i$, $A_i$ and $B_i$ are continua and $X = A_i \cup B_i$

2) $A_m \cap A_n \neq \emptyset$ if, and only if, $A_{m+1} \cap A_{n+1} \neq \emptyset$

3) $B_m \cap B_n \neq \emptyset$ if, and only if, $B_{m+1} \cap B_{n+1} \neq \emptyset$.

Applying [5, Lemma 2], we conclude that either $\cap \{A_n : n > 0\} \neq \emptyset$ or $\cap \{B_n : n > 0\} \neq \emptyset$. Let $L = \cap \{A_n : n > 0\}$ and suppose that $L \neq \emptyset$. Since X is hereditarily unicoherent, $L$ is a continuum. Also, $L \subseteq A_0$, so $L$ is a proper subcontinuum of X. Clearly, $h[A_i] \subseteq A_{i-1}$ if $i > 1$, so $h[L] \subseteq L$, contradicting the fact that no proper subcontinuum of X is mapped into itself. This concludes the proof.

References

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University of Kentucky
Lexington, Kentucky 40506

and

The University of Alabama at Birmingham

Birmingham, Alabama