STEENROD HOMOTOPY THEORY,
HOMOTOPY INDEMPOTENTS, AND
HOMOTOPY LIMITS

by

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1. Introduction

In 1940, N. E. Steenrod [26] introduced a homology theory $SH_\ast$ on compact metric pairs, which is exact on all pairs $(X,A)$. The continuity axiom of Čech homology is replaced by a short-exact sequence ([26], and J. Milnor [23]):

\[(1.1) \ 0 \to \lim_{n} \{H_{i+1}(X_n) \to SH_i(X) \to \check{H}_i(X) \to 0.\]

In (1.1), $\{X_n\}$ is any tower (inverse sequence) of polyhedra whose inverse limit is $X$. D. A. Edwards and the author [11, Ch. VIII] observed that any generalized homology theory yields a "Steenrod" homology theory on the category of towers of spaces; in fact, on Grothendieck's category pro-Top of inverse systems of spaces. See M. Artin and B. Mazur [1, Appendix] for pro-Top. Our joint work required a strong (Steenrod) homotopy theory of pro-spaces [11, Ch. III]. Although the precise definition of Steenrod homotopy theory is fairly complex, we can relate Čech (Artin-Mazur, [4]) and Steenrod [11, 12, 13] homotopy theory in §2 below. Motivated by the Brown-Douglas-Fillmore [2,3] theory of normal operators, D. S. Kahn, J. Kaminker and C. Schochet gave a different, independent development of generalized Steenrod homology theories [16, 17].

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The rest of this paper is organized as follows. §§3-5 survey Steenrod homotopy theory. Homotopy limits, largely following [11, Ch. IV] are described in §3. §4 recalls the Edwards-Geoghegan [10] result that "idempotents split in pro-categories." J. Dydak and P. Minc [8], and P. Freyd and A. Heller [14] independently obtained a non-split idempotent in unpointed homotopy theory. Dydak observed an important consequence in pro-homotopy: a map which is an equivalence in Čech homotopy theory but not in Steenrod homotopy theory. We summarize these results in §5 to complete the relation between Čech and Steenrod homotopy theory.

Finally, we give "geometric" (Artin-Mazur [1] type) formulations of a coherent completion functor (§6) and a strong shape functor (§7). We conclude with a "dual" construction of a coherent Quillen [26] \( + - \) construction in §8.

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2. Čech and Steenrod Homotopy Theory

T. A. Chapman's [5] beautiful complement theorem relating the shape theory of compacta \( K \) in the pseudo-interior \( s = \prod_{i=1}^{\infty} (-1,1) \) of the Hilbert cube \( Q = \prod_{i=1}^{\infty} [-1,1] \) and the homeomorphism type of \( Q \setminus K \) has the following corollary [5]: a category isomorphism between the shape category of such compacta and the weak proper homotopy category of their complements. Later, Edwards and the author [11, pp. 228-232] obtained a similar relationship between strong shape theory [11, especially Ch. VI and VIII] and the more geometric proper homotopy theory, which together with Chapman's correspondence
homotopy theory, which together with Chapman's correspondence yields a commutative square

\[
\begin{array}{ccc}
\text{strong shape} & \xrightarrow{\pi} & \text{shape} \\
\text{(Steenrod homotopy)} & & \text{(Čech homotopy)} \\
\text{category of compacta} & & \text{category of compacta} \\
K \subset s \subset Q & & K \subset s \subset Q \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{proper homotopy category of complements} & \xrightarrow{\pi'} & \text{weak proper homotopy category of complements} \\
\{Q \setminus K, K \subset s \subset Q\} & & \{Q \setminus K, K \subset s \subset Q\} \\
\end{array}
\]

In diagram (2.1), Steenrod homotopy theory refers to the strong homotopy theory of inverse systems $\text{Ho}(\text{pro-Top})$ of Edwards and the author [11, especially Ch. VIII]; Čech homotopy theory to the Artin-Mazur theory [1] $\text{pro-Ho}(\text{Top})$. The vertical map $\phi$ is Chapman's isomorphism cited above; similarly, $\phi'$ is the isomorphism of [11, \textit{loc. cit.}]. The maps $\pi$ and $\pi'$ are natural quotient maps. The Čech nerve $\text{Top} \rightarrow \text{pro-Ho}(\text{Top})$ yields shape theory (see, e.g. Edwards [9]); a Vietoris functor $\text{Top} \rightarrow \text{Ho}(\text{pro-Top})$ (T. Porter [24]) yields strong shape theory [11]. The distinction between Čech and Steenrod homotopy theory was first recognized by D. Christie [7], although he lacked D. Quillen's abstract homotopy theory [25] needed to define $\text{Ho}(\text{pro-Top})$ [11]. We shall give a more "geometric" version of strong shape theory (still using [11]) in §7---some of whose properties were obtained in a conversation with Kozlowski and Liem. Details and applications will be described elsewhere. \textit{Added in proof.} See joint work with A. Calder [30]. J. Dydak and J. Segal [31] and Y, Kodama and J. Ono [32] recently gave independent equivalent descriptions of strong shape theory.

Although the relationship between $\text{Ho}(\text{pro-Top})$ and pro-
Ho (Top) appears quite complicated [11], useful results are available for towers (countable inverse systems). Let Top* be the category of pointed spaces and maps. In 1974, J. Grossman [15], and Edwards and the author [11, Theorem (5.2.1)] independently proved the following.

(2.2) Theorem. Let \( \{X_m\} \) and \( \{Y_n\} \) be towers of pointed spaces. Then there is a short-exact sequence of pointed sets.

\[
\ast \to \lim_{n=1} \colim_{m=1} \left\{ \Sigma X_m, Y_n \right\} \to \text{Ho}(\text{towers-Top}^*)\left(\{X_m\}, \{Y_n\}\right) \to \ast.
\]

The functor \( \pi \) is also onto in unpointed pro-homotopy.

The appropriate derived functor \( \lim^1_n \) for towers of (non-abelian) groups was defined by Bousfield and Kan [4, p. 251]. Chapman and L. Siebenmann [6] asked whether every weak-proper-homotopy-equivalence is a proper-homotopy-equivalence. A useful partial answer appears in [11, Theorem (5.2.9)]; similar results hold for pointed spaces [11, loc. cit.], and for proper homotopy [12].

(2.3) Theorem [11]. Let \( f: \{X_m\} \to \{Y_n\} \) be a map in \( \text{Ho} (\text{towers-Top}) \) which is invertible in \( \text{towers-Ho} (\text{Top}) \). Then there is an isomorphism \( g: \{X_m\} \to \{Y_n\} \) in \( \text{Ho} (\text{towers-Top}) \) with \( g \) equivalent to \( f \) in \( \text{towers-Ho} (\text{Top}) \).

(2.4) Corollary [11, Corollary 5.2.17]. The isomorphism classification problems in \( \text{Ho} (\text{towers-Top}) \) and \( \text{towers-Ho} (\text{Top}) \) are equivalent.

(2.5) Caution: non-equivalent maps in \( \text{Ho} (\text{towers-Top}) \) may become equivalent in \( \text{towers-Ho} (\text{Top}) \).
Dydak [8] recently observed that the map $f$ of (2.3) need not itself be invertible in $\text{Ho} (\text{towers-Top})$. This result involves homotopy limits (§3) and splitting idempotents (§4), and will be discussed in §5.

3. Homotopy Limits

It is easy to see that even towers do not have limits in homotopy theory. D. Puppe gave the following example in a 1976 lecture in Dubrovnik. Let

$$K = \{K(Z,2) \overset{3}{\rightarrow} K(Z,2) \overset{3}{\rightarrow} \cdots \},$$

where "3" denotes a degree 3 map. Suppose that $K$ had a limit $\overline{K}$ in $\text{pro-} \text{Ho} (\text{Top})$. Then the Barratt-Puppe sequence

$$s^2 \overset{2}{\rightarrow} s^2 + c + s^3 + \cdots$$

would yield an exact sequence

$$\begin{array}{cccc}
[S^3, K] & \longrightarrow & [C, K] & \longrightarrow & [S^2, K] \\
\| & & \| & & \|
\lim\{[S^3, K(Z,2)], 3\} + \lim\{[C, K(Z,2)], 3\} + \lim\{S^2, K(Z,2)\}, 3\}
\| & & \|
0 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 0,
\end{array}$$

an obvious contradiction. However, homotopy limits exist in $\text{Ho} (\text{pro-Top})$; for a pro-space $Y = \{Y_\alpha\}$, the functor

$$\text{Ho} (\text{pro-Top}) (-, \{Y_\alpha\})$$
on $\text{Top} \subset \text{pro-Top}$ is represented by $\text{holim}\{Y_\alpha\}$ (Edwards and the author [11, Ch. IV]):

$$\text{Ho} (\text{pro-Top}) (X, \{Y_\alpha\}) = \text{Ho} (\text{Top}) (X, \text{holim}\{Y_\alpha\}).$$

The construction of [11], reminiscent of J. Milnor's [20] mapping telescope, consists of replacing $\{Y_\alpha\}$ by a fibrant object (using S. Mardešić [17], and [11]) $Y_\beta'$ and applying the ordinary inverse limit to $Y_\beta'$. Other constructions were given by A. K. Bousfield and D. M. Kan [4, Ch. X],
and R. Vogt [29].

4. Splitting Homotopy Idempotents

D. A. Edwards and R. Geohegan, in their work [10] on a Wall obstruction on shape theory, showed that "idempotents split in pro-categories." Let $r: X \to X$ be a homotopy idempotent, i.e., $r^2 = r$. If there is a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{d} & Y \\
\downarrow{u} & & \downarrow{d} \\
X & \xrightarrow{u} & Y
\end{array}
\]

with $du = id_Y$ and $ud = r$, then $r$ is said to split. Let $Y$ be the tower

\[ Y = \{X \overset{r}{\to} X \overset{r}{\to} X \overset{r}{\to} \cdots \}. \]

Then $r$ induces maps $X \xrightarrow{d} Y$ in Čech homotopy theory (towers-$Ho(Top)$)

\[
\begin{array}{ccc}
X & \xrightarrow{d} & Y \\
\downarrow{u} & & \downarrow{d} \\
X & \xrightarrow{u} & X \xrightarrow{r} X \xrightarrow{r} \cdots
\end{array}
\]

(all arrows $r$)

which split $r$. We may replace $Y$ by a tower of fibrations, and then replace $u$ and $d$ by strict maps (maps in Steenrod homotopy $Ho(towers-Top)$) [10], see also [11]. Suppose $du = id_Y$ in $Ho(towers-Top)$. Then $r$ splits in $Ho(towers-Top)$ [10] because $holim$ is a functor:

\[ X \simeq holim X \xrightarrow{holim d} \xleftarrow{holim u} holim Y. \]

The Dydak-Minc [8], Freyd-Heller [14] example of a non-split idempotent in unpointed homotopy (described in §5) thus shows that a weak equivalence need not be a strong equivalence [8], compare Theorem (2.3) (Edwards and the author), above.
5. The Dydak-Minc-Freyd-Heller Example [8, 14]

Let $G$ be the group

$$\langle g_1, g_2, \ldots | g_i^{-1} g_j g_i = g_{j+1}, i < j \rangle.$$ 

Let $f: G \to G$ be the monomorphism defined by

$$f(g_i) = g_{i+1}.$$ 

Then $f^2(g) = g_1^{-1} f(g) g_1$, so that $f$ is conjugate to $f$, and the induced map

$$r = K(f, 1): K(G, 1) \to K(G, 1)$$ 

is an unpointed homotopy idempotent [8, 14]. Dydak gives a straight-forward argument that $r$ does not split—we sketch his argument here. If $r$ splits, $r$ splits through a $K(H, 1)$. In the resulting diagram

$$K(G, 1) \xrightarrow{d} K(H, 1) \xleftarrow{u}$$

d is both mono and epi on $\pi_1$ by construction, hence $d$ is a homotopy equivalence by the Whitehead theorem. This implies $\text{Im} f = G$, an evident contradiction.

Freyd and Heller [14] have obtained a wealth of interesting results about $G$.

6. Pro-Finite Completions

Artin and Mazur [1] introduced the following pro-finite completion in order to prove comparison theorems in étale homotopy theory. Let $Y$ be a finite, pointed CW complex. The pro-finite completion of $Y$, $\hat{Y}$, is the category whose objects are (homotopy classes of pointed) maps

$$Y \to \hat{Y}_g,$$ 

with $\pi_i(\hat{Y}_g) = \begin{cases} 0, \text{almost all } i \\ \text{finite, otherwise,} \end{cases}$

and whose morphisms are homotopy-commutative diagrams.
This yields a completion functor $\hat{\cdot}: \text{Ho (finite pointed complexes)} \to \text{pro-Ho (Top)}$ as follows. Given a map $X \to Y$, associate to each object $Y \to Y_\alpha$ in the completion $\hat{Y}$ of $Y$ the composite map $X \to Y + Y_\alpha$, an object in $\hat{X}$. This yields a map $\hat{f}: \hat{X} \to \hat{Y}$, see [1, Appendix].

D. Sullivan [28] showed that $\text{pro-Ho (Top)}(\cdot, Y)$ is representable, that is,

$$\text{pro-Ho (Top)}(\cdot, Y) = [\cdot, Y].$$

Later, A. K. Bousfield and D. M. Kan [4, Ch. I] introduced a different, rigid, completion, the R-completion $\{R_y\}$, and observed that $\{R_y\}$ is cofinal in an Artin-Mazur type R-completion. Here $R$ is a commutative ring with identity; we call $\{R_y\}$ rigid because the construction of $\{R_y\}$ yields a functor into $\text{Ho (pro-Top)}$.

In developing the "genetics of homotopy theory" [28], D. Sullivan remarked that a simple rigid completion functor could prove useful. We shall rigidify (i.e., lift to $\text{Ho (gpro-Top)}$) the Artin-Mazur completion functor by a simple trick. Objects of $\text{gpro-Top}$ are inverse systems of spaces which are filtering up to homotopy. See [30].

(6.2) **Definition.** The rigid pro-finite completion of $Y$ is the category $\hat{Y}_{\text{rig}}$ whose objects are pointed maps

$$Y \to Y_\alpha', \text{ with } \pi_i(Y_\alpha') = \begin{cases} 0, & \text{almost all } i \\ \text{finite}, & \text{otherwise}, \end{cases}$$
and whose morphisms are \textit{strictly commutative diagrams}

\[
\begin{array}{ccc}
Y & \xrightarrow{\text{fibration}} & Y_{\beta} \\
Y_{\alpha} & \downarrow \text{fibration} & \quad Y_{\gamma} \\
\end{array}
\]

in which the bottom map is a \textit{fibration}.

Because the pullback of a diagram

\[
\begin{array}{ccc}
Y_{\alpha} & \xrightarrow{\text{fibration}} & Y_{\gamma} \\
\downarrow \text{fibration} & & \downarrow \text{fibration} \\
Y_{\beta} & \quad & Y_{\gamma} \\
\end{array}
\]

is also a "homotopy pullback," \(\hat{Y}_{\text{rig}}\) has weak equalizers. It follows easily that \(\hat{Y}_{\text{rig}}\) is filtering up to homotopy.

Further, the functor

\[
\text{Ho} \left(\text{pro-Top}\right) \left[\cdot, \hat{Y}_{\text{rig}}\right] \equiv \left[\cdot, \text{holim} \hat{Y}_{\text{rig}}\right]
\]

is clearly representable, and the Bousfield-Kan spectral sequence [3, Ch. XI] shows that \(\text{holim} \hat{Y}_{\text{rig}} = \overline{Y}\), see (6.1).

\textbf{(6.3) Remarks.} The rigid pro-finite completion \(\hat{\cdot}_{\text{rig}}\) induces a \textit{reflection} \(\text{Ho}(\text{gpro-Top}) \rightarrow \text{Ho}(\text{gpro-Top})\), i.e.,

\[
(\hat{\cdot}_{\text{rig}})^{\text{rig}} = \hat{\cdot}_{\text{rig}}, \text{ always.}
\]

In contrast, for the Bousfield-Kan completion, \(\mathbb{Z}_{\omega}\text{RP}^2\) and \((\mathbb{Z}_{\omega})^2\text{RP}^2\) are not equivalent, thus, \(\text{RP}^2\) is called \(\mathbb{Z}\)-bad [4, Ch. I]. Further, there should be an induced homotopy theory (closed model structure [25]) on the image of \(\hat{\cdot}_{\text{rig}}\) under which \(\hat{\cdot}_{\text{rig}}\) preserves fibration and cofibration sequences. Sullivan's completion functor cannot preserve both types of sequences [28]. Note however, that the inverse limit \(\text{lim} \left(\text{gpro-Top} \rightarrow \text{Top}\right)\) preserves fibration sequences but not cofibration sequences.

\textbf{7. Strong Shape Theory}
S. Mardešić [21] introduced the following Artin-Mazur approach to the shape theory. The shape of a topological space $X$, $\text{sh}(x)$, is the category whose objects are homotopy classes of maps $X \rightarrow X_\alpha$, where $X_\alpha$ is an ANR, and whose morphisms are homotopy-commutative triangles of the form (6.1). Let $f: X \rightarrow Y$ be a continuous map. Then each map $Y \rightarrow Y_\alpha$ in $\text{sh}(Y)$ induces a map $X \rightarrow Y_\alpha$ by composition with $f$; this yields a shape functor

$$\text{sh}: \text{Top} \rightarrow \text{pro-Ho(ANR)} \subset \text{pro-Ho(Top)}.$$ 

One can replace "ANR" by "polyhedron" (possibly infinite) in the Mardešić definition.

We rigidify the Mardešić shape functor (i.e., lift $\text{sh}$ to $\text{Ho (pro-Top)}$) by a trick analogous to (6.1), and briefly describe the resulting geometric strong shape theory. A Vietoris functor approach to strong shape theory is developed in Porter [24] and [11, Ch. VIII].

(7.1) Definition. The strong shape of a topological space $X$, $s = \text{sh}(X)$, is the category whose objects are maps $X \rightarrow X_\alpha$, with $X_\alpha$ a polyhedron, and whose morphisms are strictly commutative triangles

\[
\begin{array}{ccc}
X & \rightarrow & X_\alpha \\
\downarrow & & \downarrow \text{PL} \\
X & \rightarrow & X_\beta
\end{array}
\]

in which the bottom map is PL.

(7.2) Proposition. This construction yields a functor $s = \text{sh}: \text{Top} \rightarrow \text{pro-(polyhedra)} \subset \text{pro-Top}$. Further, the composite functor $\pi \circ s = \text{sh}: \text{Top} \rightarrow \text{pro-Ho(Top)}$ is equivalent
to Mardešić’s shape functor sh.

Proof. Observe that the equalizer of two PL maps of polyhedra is a polyhedron. Thus $s - \sh(X)$ has equalizers. The rest is easy and omitted.

(7.3) Proposition. The functor $s - \sh$ induces a functor on homotopy categories

$$s - \sh: \Ho(Top) \rightarrow \Ho(pro-Top).$$

Proof. Let $H: X \times I \rightarrow Y$ be a homotopy, with $H_0 = f$ and $H_1 = g$. Form the commutative diagram in $pro$-Top

$$
\begin{array}{ccc}
\sh(X \times 0) & \rightarrow & \sh(f) \\
\downarrow & & \downarrow \\
\sh(X \times I) & \rightarrow & \sh(H) \\
\downarrow & & \downarrow \\
\sh(X \times I) & \rightarrow & \sh(g) \\
\end{array}
$$

Each map $\phi_\alpha: X \times I \rightarrow Z_\alpha$ in $s - \sh(X \times I)$ factors as

$$X \times I \xrightarrow{(\phi_\alpha', \proj_1)} Z_\alpha \times I \xrightarrow{\proj} Z_\alpha',$$

hence the map $s - \sh(X \times 0) \rightarrow s - \sh(X \times I)$ is represented by the inverse system of maps

$$
\begin{array}{ccc}
X \times 0 & \rightarrow & X \times I \\
\downarrow & & \downarrow \\
\{Z_\alpha \times 0\} & \rightarrow & \{Z_\alpha \times I\}
\end{array}
$$

Thus the map $s - \sh(X \times 0) \rightarrow s - \sh(X \times I)$ is a trivial cofibration (i.e. cofibration and equivalence in $\Ho(pro$-Top)), similarly for $X \times 1$. (Note: we are not asserting that the maps $X \times I \rightarrow Z_\alpha \times I$ factor as $g \times \id$, only that the bonding maps $Z_\alpha \times I \rightarrow Z_\beta \times I$ factor in this way). The conclusion follows.

We now restrict the domain of $s - \sh$ to the category $CM$ of compact metric spaces. We may then assume $s - \sh$ takes
values in pro-(finite polyhedra).

(7.4) Proposition. For X in CM, X ≅ lim ∗ s - sh(X).

Proof. It suffices to prove that natural map

p: X + lim ∗ s - sh(X) is bijective. Because any two distinct
points of X are separated by a map of X into [0,1], p is in­
jective. Further any map X + Xα in s - sh(X) which misses a
point * in Xα factors through a subpolyhedron Xα' ⊂ Xα with
* ∉ Xα'. The conclusion follows.

Propositions (7.2)-(7.4) are summarized in the follow­
ing diagram--which justifies calling s - sh a strong-shape
functor--

\[
\begin{array}{c}
CM \xrightarrow{\text{lim}} \text{pro-(finite polyhedra)} \\
\downarrow \hspace{2cm} \downarrow \\
\text{Ho(CM)} \xrightarrow{s - sh} \text{Ho(pro-(finite polyhedra))} \\
\hspace{2cm} \downarrow \\
\text{sh} \hspace{2cm} \text{pro-Ho(finite polyhedra)}
\end{array}
\]

(7.5) Proposition (with Kozlowski-Liem). For any com­
pact metric pair (X,A), the sequence

s - sh(A) + s - sh(X) + s - sh(X/A)

is a cofibration sequence in pro-top.

Proof. Consider the inverse system whose objects are

\[
\begin{array}{c}
A \xrightarrow{\text{X}} X \xrightarrow{\text{X/A}} \\
\downarrow \hspace{2cm} \downarrow \hspace{2cm} \downarrow \\
A_\alpha \xrightarrow{\text{X_\alpha}} X_\alpha \xrightarrow{\text{X_\alpha/A_\alpha}}
\end{array}
\]

with (X_\alpha,A_\alpha) a finite polyhedral pair, and whose bonding
maps are defined analogously with (7.1). The induced systems

\{X_\alpha\} and \{X_\alpha/A_\alpha\} are clearly cofinal in s - sh(X) and
s - sh(X/A), respectively: given X + X₂ in s - sh(X), let
A₂ = X₂, and given X/A + P₂ in s - sh(X/A), let A₂ be a
point, and let X₂ = P₂.

Finally Kozlowski remarked that any solid-arrow diagram
\[
\begin{array}{ccc}
A & \rightarrow & P \\
\downarrow & & \downarrow \\
X & \rightarrow & CP
\end{array}
\]
(where CP is the cone on P) with (X,A) a compact metric pair
admits a filler (compare Kuratowski's extension lemma for
Čech nerves [18, p. 122]). This implies that the induced
inverse system A is cofinal in s - sh(A). The conclusion
follows.

Propositions (7.2) and (7.5) imply the following (com­
pare D. A. Edwards and the author [11, Ch. VIII]).

(7.6) Proposition. For any homology theory \( h \), on
pro-Top, the composite \( h \circ s - sh \) is a homology theory on
CM.

By comparing s - sh with the Vietoris functor [24],
and using the machinery of [11, Ch. VIII], we can prove the
appropriate continuity formula

\[
(7.7) \quad s - sh(\lim(X_n)) = \lim s - sh(X_n) \text{ in } Ho-pro\ (Top) \text{ for } s - sh. \quad \text{Formula (7.7) implies the following.}
\]

(7.8) Proposition (compare [11, Theorem (8.2.21)])
The composite functor \( h \circ s - sh \) is a generalized Steenrod
homology theory.

(7.9) Remarks. (a) Formula (7.7) is analogous to the
Steenrod-Milnor short exact sequence (1.1).
(b) The relationship between the strong shape category and the shape category is analogous to the relation between Steenrod and Čech homotopy theory, see (2.2)-(2.5), above.


Unfortunately we do not have a purely geometric proof of (7.7).

8. A Rigid + - Construction

We outline a rigidification of Quillen's + - construction [26] using techniques of Edwards and the author [11], dual to §§3-7, above. For simplicity, let $\text{Top}_p$ be the category of pointed spaces with perfect fundamental groups. We define a functor

$$+ : \text{Top}_p \rightarrow \text{Top}_p$$

such that our $X^+$ is equivalent to Quillen's $X^+$, and such that the diagram

$$\begin{array}{ccc}
\text{Top}_p & \xrightarrow{+ (\text{ours})} & \text{Top}_p \\
\downarrow & & \downarrow \\
\text{Ho} (\text{Top}_p) & \xrightarrow{+ (\text{Quillen})} & \text{Ho} (\text{Top}_p)
\end{array} \quad (8.1)$$

commutes. In fact our techniques work for pairs $(X,H)$ where $X$ is a pointed space and $H$ a normal subgroup of $\pi_1(X)$ containing $[\pi_1(X),\pi_1(X)]$.

First associate to $X$ the category $+ (X)$ whose objects
are maps \( X \rightarrow X_\alpha \) with and whose morphisms are commutative triangles

\[
\begin{array}{ccc}
X & \longrightarrow & X_\alpha \\
\downarrow & & \searrow \text{cofib.} \\
X_\alpha & \longrightarrow & X_\beta
\end{array}
\]

in which the bottom map is a cofibration. It is easy to check that \( + (X) \) is a direct system, filtering up to homotopy (see [30], reverse the arrows in the Artin-Mazur definition [1, Appendix] of an (inverse) filtering category).

Next define

\[
(8.2) \quad X^+ = \text{hocolim} (+ (X))
\]

where hocolim is the homotopy colimit ([11, pp. 169-171] the dual of the homotopy limit sketched in §3 or Bousfield-Kan [4, Ch. XII]). It is easy to check that Definitions (8.2)-(8.3) yield the required properties. Details are omitted.

References

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