THE STONE-ČECH COMPACTIFICATION
AND SHAPE DIMENSION

by

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Introduction

This paper is a continuation of the work of Keesling [8] and Keesling and Sher [9] in studying the shape properties of the Stone-Čech compactification of a space. For a compact space \( X \), shape theory is the study of those properties of \( X \) that can be deduced by considering the homotopy classes of mappings of \( X \) into various polyhedra \( P, [X,P] \). Here we are concerned with \( P = M^n \), a closed manifold which is a \( K(\pi,1) \) and also \( P = S^n \times S^1 \). By considering maps of the Stone-Čech compactification onto these polyhedra we are able to draw certain conclusions about the shape dimension of the Stone-Čech compactification. Our specific results are as follows.

Let \( f : X \rightarrow Y \) be a map. Then \( f \) is homotopically onto provided that if \( g : X \rightarrow Y \) is homotopic to \( f \), then \( g \) is onto \( Y \). Let \( n \geq 1 \) be a fixed integer and let \( M^n \) be a closed manifold having the property that the universal covering space for \( M^n \) is \( R^n \). Suppose that \( X \) is locally compact and \( \sigma \)-compact such that for every compact set \( K \subseteq X \), there is a compact set \( L \subseteq X-K \) such that \( \dim L \geq n \). Then there is a map \( f : \beta X \rightarrow M^n \) which is homotopically onto. Also \( f|_{\beta X-X} \) is homotopically onto \( M^n \) as well. This implies that the shape dimension of \( \beta X \) is at least \( n \) and that the shape dimension of \( \beta X-X \) is at least \( n \) also. If \( X \) is a Lindelöf space and \( K \) is a compactum contained in \( \beta X-X \), then if \( \dim K \geq n \) and \( M^n \) is as above, then
there is a map $f: K \to M^n$ which is homotopically onto. This implies that for compacta $K \subseteq \beta X-X$, dimension and shape dimension coincide. This is generally not true since the dimension of the closed unit ball $B^n$ in $\mathbb{R}^n$ is $n$, but the shape dimension of $B^n$ is 0.

The above results are used to show the following. For every $n \geq 2$, there is a polyhedron $P_n$ of dimension $n$ and a map $f: P_n \to T^n$ to the $n$-dimensional torus $T^n$ such that $f$ is homotopically onto and $f^*: H^k(T^n) \to H^k(P_n)$ is the zero homomorphism for all $k \geq 2$. Also for every $n \geq 2$, there is a metric continuum $X_n$ such that the shape dimension of $X_n$ is $n$ and $\dim X_n = n$, but the shape dimension of the suspension $\Sigma X_n$ is equal to 2. This last example answers a question of S. Nowak [11].

We then consider maps onto $S^n \vee S^1$ and obtain similar results. However, in this case we encounter a dimension theoretic difficulty and the results are not quite so general as are the results about mappings onto manifold $K(n,1)$'s.

It appears from the results in this paper and in [8] and [9] that shape theory can be a useful tool in studying the Stone-Čech compactification. The techniques and results will likely have wider application in the future.

Preliminaries

The reader is assumed to be familiar with shape theory. The paper by S. Mardešić is a good reference [12]. We will only be concerned about compact spaces in this paper, however, and thus will not need the full generality of [12] which develops shape theory for all topological spaces. The
reader is assumed to be familiar with the basic properties of the Stone-Čech compactification. Either Gillman and Jerison [2] or Walker [14] would be a good reference. In dimension theory we need only the basic results which we quote from [4] and the Appendix of [10]. We now state the most important of these for easy reference. By dim X we mean the Lebesgue covering dimension.

0.1. **Theorem** ([4, 19, p. 85]). The dimension of a normal space X is the supremum of the integers n such that there is an essential mapping onto \( I^n \).

Here an essential mapping \( f: X \to I^n \) is a map such that \( f|f^{-1}(\text{Bd}I^n) \) cannot be extended to all of X with values in \( \text{Bd}I^n \). We will also need the Sum Theorem for dimension.

0.2. **Theorem** ([4, Theorem 7, p. 148]). If X is a normal space with \( X = \bigcup_{i=1}^{\infty} X_i \) with \( X_i \) closed for all \( i \), then \( \dim X = \sup\{\dim X_i\} \).

We will also need the Hopf Extension Theorem.

0.3. **Theorem** ([10, 36-17, p. 207]). Let X be a paracompact space of covering dimension \( \leq n+1 \) and A a closed subset of X. Let \( f: A \to S^n \). Then f is extendable over X if and only if \( f^*(H^n(S^n)) \subset i^*(H^n(X)) \) where \( i: A \to X \) is the inclusion map.

If X is a compact space, then the shape dimension of X is the least integer n such that there is a space Y with \( \dim Y = n \) such that Y shape dominates X. We will denote this by \( \text{SD} X = n \). If X is a compact metric space, then this is
the same as K. Borsuk's fundamental dimension, $F_d X$. Results from [11] will be used concerning fundamental dimension. We will now state and prove a useful elementary result about shape dimension. (For a stronger result see the discussion preceding Lemma 2.3 of [15] by J. Dydak.)

0.4. Theorem. Let $X$ be a compact space with $S_d X < n$. Let $f : X \to P$ be a map with $P$ a finite polyhedron. Then there is a finite polyhedron $Q$ with $\dim Q \leq n$ and maps $g : X \to Q$ and $r : Q \to P$ such that $r \circ g$ is homotopic to $f$.

Proof. Let $Y$ shape dominate $X$ with $\dim Y = n$. Let $f : X \to P$ be a map. Let $F : X \to Y$ and $G : Y \to X$ be shape morphisms with $G \circ F = S(\text{id}_X)$. Let $f' : Y \to P$ be a map such that $S(f') = S(f) \circ G$. Because $Y$ is $n$-dimensional, there is an open cover $U$ of $Y$ of order at most $n+1$ such that the barycentric map $g_U : Y \to N(U)$ to the nerve of $U$ factors the map $f'$ up to homotopy. That is, there is a map $r : N(U) \to P$ such that $r \circ g_U$ is homotopic to $f'$. Then $Q = N(U)$ will have dimension at most $n$. Let $g : X \to Q$ be a map such that $S(g) = S(g_U) \circ F$. Then $S(r \circ g) = S(r) \circ S(g) = S(r) \circ S(g_U) \circ F = S(f') \circ F = S(f) \circ G \circ F = S(f)$. Thus $S(r \circ g) = S(f)$. This implies that $r \circ g$ and $f$ are homotopic since $P$ is an ANR.

We let $H^n(X,A)$ denote $n$-dimensional Čech cohomology with integer coefficients where $X$ is a paracompact space and $A$ is a closed subset of $X$. We assume a basic knowledge of Čech cohomology. The following theorem follows from the Hopf Extension Theorem.

0.5. Theorem. If $X$ is a paracompact space of covering
dimension n and A is a closed subset of X, then \( H^n(X, A) \) is isomorphic to \([(X, A), (S^n, p)]\).

1. Maps onto Manifold \( K(n, 1)'s \)

Let \( M^n \) be a closed n-manifold whose universal covering space is \( \mathbb{R}^n \). For example, \( M^n \) could be the n-dimensional torus \( T^n \). However, many more possibilities exist, see for instance [5]. It is an unsolved problem whether the covering space of a closed n-manifold \( K(n, 1) \) must be \( \mathbb{R}^n \) [3, §3, p. 423]. For \( n = 3 \), this is related to the Poincaré Conjecture.

Our proofs seem to require that the universal covering space for \( M^n \) be \( \mathbb{R}^n \) and so we make this assumption rather than that \( M^n \) be a closed manifold \( K(n, 1) \). In this first section of the paper we study maps from \( \beta X \) onto \( M^n \). The results give us a better understanding of the shape of \( \beta X \) and of compacta contained in \( \beta X - X \). The main results in this section are Theorems 1.2, 1.3, 1.6, and 1.8. In the next section we give some further applications of these theorems. First we prove an important lemma.

1.1. Lemma. Suppose that \( P \) is a finite polyhedron and that \( H: \beta X \times I + P \) is a homotopy. Suppose that \( \tilde{P} \) is the universal covering space for \( P \). Let \( K \) be a compact set in \( \tilde{P} \). Then there is an open set \( U \) in \( \tilde{P} \) containing \( K \) such that the closure of \( U \) in \( \tilde{P} \) is compact and such that for each \( x \in X \) if \( h: \{x\} \times I + \tilde{P} \) is any lift of the path \( H|\{x\} \times I \), then if \( h(\{x\} \times I) \cap K \neq \emptyset \), then \( h(\{x\} \times I) \subset U \).

Proof. Suppose not. Suppose it is false for the compact set \( K \subset \tilde{P} \). Let \( \{U_i\}_{i=1}^{\infty} \) be a sequence of open sets in \( \tilde{P} \) such that (1) \( K \subset U_1 \), (2) \( \overline{U_i} \subset U_{i+1} \) for all \( i \), (3) \( \overline{U_i} \) compact for
all $i$, and (4) $\tilde{P} = \bigcup_{i=1}^{\infty} U_i$. Since the lemma is assumed false for $K$, for each $i$ there is a point $x_i \in X$ such that some lift $h_i$ of $H|\{x_i\} \times I$ has the property that $h_i(\{x_i\} \times I) \cap K \neq \emptyset$ and $h_i(\{x_i\} \times I) \cap (\tilde{P} - U_i) \neq \emptyset$. Let $t_i(1) \in I$ be such that $h_i(x_i, t_i(1)) \in K$ and $t_i(2) \in I$ be such that $h_i(x_i, t_i(2)) \notin U_i$. Then let $x \in \beta X$ be a limit point of the set $\{x_i\}_{i=1}^{\infty}$. Let $\{i_n\}$ be a subnet of the positive integers such that (1) $x_{i_n} \to x$, (2) $t_{i_n}(1) \to t(1) \in I$, and (3) $t_{i_n}(2) \to t(2) \in I$. Such a net and points $t(1)$ and $t(2) \in I$ can be found routinely and we omit the details. Now let $O$ be a contractible open set in $\tilde{P}$ containing $H(x, t(1))$ such that the covering map $c: \tilde{P} \to P$ has the property that $c$ restricted to each component of $c^{-1}(O)$ is a homeomorphism onto $O$. By the continuity of $H: \beta X \times I \to P$, there must be a $\beta$ such that for all $\alpha \geq \beta$, $H(\{x_{i_n}\} \times I_{i_n}) \subset O$ where $I_{i_n}$ is the subinterval of $I$ joining $t_{i_n}(1)$ to $t(1)$. So we may suppose that $H(\{x_{i_n}\} \times I_{i_n}) \subset O$ for all $\alpha$. Now $K$ is compact in $\tilde{P}$ and since the components of $c^{-1}(O)$ form a discrete set in $\tilde{P}$, there can be at most a finite number of these components which intersect $K$. Call these components $\{O_1, \ldots, O_k\}$. For one of these components $O_j$ there must be a subnet of $\{i_n\}$ such that $h_{i_n}(x_{i_n}, t_{i_n}(1)) \in O_j$ for all $i_n$ of the subnet. By renaming the subnet let us assume that $h_{i_n}(x_{i_n}, t_{i_n}(1)) \in O_j$ for all $\alpha$. Let $p: O \to O_j$ be the inverse of $c|O_j$. Then let $B = \{x_{i_n}\} \cup \{x\} \subset \beta X$ and let $f: B \to \tilde{P}$ be defined by $f \equiv p\circ H|B \times \{t(1)\}$. Then $f$ is a continuous lift for $H|B \times \{t(1)\}$. Thus there is a unique lift of $H|B \times I$, $\tilde{H}: B \times I \to \tilde{P}$, such that $\tilde{H}|B \times \{t(1)\} \equiv f$. Now by the definition of $f = \tilde{H}|B \times \{t(1)\}$, $\tilde{H}(x_{i_n}, t(1)) \in O_j$ for all $\alpha$. Also $H(\{x_{i_n}\} \times I_{i_n}) \subset O$ for all
a. Thus $\tilde{H}(x_i^a, t_i^a(1)) \in O_j$ also. Thus $\text{Ext}_{i}(x_i^a) \times I \equiv h_i^a$ for all $a$. Now $\tilde{H}(x, t(2)) \in U_i$ for some $i$. But since $\tilde{H}$ is continuous, there must be a $\beta$ such that for all $a \geq \beta$,

$\tilde{H}(x_i^a, t_i^a(2)) \in U_i$. There is also a $\gamma$ such that for all $a \geq \gamma$, $i_a > i$. Then let $a > \beta$ and $a > \gamma$. Then for this $a$,

$\tilde{H}(x_i^a, t_i^a(2)) \in U_i$ and $i_a > i$. This is a contradiction since $\tilde{H}(x_i^a, t_i^a(2)) = h_i^a(x_i^a, t_i^a(2)) \notin U_i$ and $U_i \subset U_i$. This contradiction proves the lemma.

We now proceed directly to the first main theorem of this section.

1.2. Theorem. Let $M^n$ be a closed $n$-manifold whose covering space is homeomorphic to $\mathbb{R}^n$. Suppose that $X$ is a locally compact $\sigma$-compact space such that for every compact set $K \subset X$, there is a compact set $L \subset X-K$ with $\dim L \geq n$.

Then there is a map $f: \mathbb{S}^X \to M^n$ which is homotopically onto.

Proof. Let $X = \bigcup_{i=1}^{\infty} K_i$ where each $K_i$ is compact and $K_i \subset \text{int} K_{i+1}$ for all $i$. Let $L_1 \subset X-K_1$ be a compact set with $\dim L_1 \geq n$. Let $L_{i+1} = X - (\bigcup_{j=1}^{i} L_j \cup \bigcup_{j=i+1}^{\infty} K_j)$ with $\dim L_{i+1} \geq n$. Then $\{L_i\}_{i=1}^{\infty}$ will be a sequence of disjoint compact subsets of $X$ with $\dim L_i \geq n$ for all $i$ and with $L_i \subset X-K_i$ for all $i$. Now if $B$ is any compact set in $X$, then there is an $i$ with $B \subset \text{int} K_i$. Thus $L_i$ will have the property that $L_i \subset X-B$. Now let $B^n$ be the closed unit ball in $\mathbb{R}^n$. Let $g_i: L_i \to B^n$ be an essential map (which is guaranteed to exist by Theorem 0.1). Then define $g: \bigcup_{i=1}^{\infty} L_i \to \mathbb{R}^n$ by $g(x) = k_i g_i(x)$ for $x \in L_i$. Now $\bigcup_{i=1}^{\infty} L_i$ is closed in $X$ and $X$ is Lindelöf, hence normal. Thus there is an extension of $g$ to all of $X$ which we also call $g$. Let $c: \mathbb{R}^n \to M^n$ be the covering map.
Then define $f$ to be the Čech extension of $c \circ g: \tilde{X} \to M^n$.

Now $g: X \to \mathbb{R}^n$ is onto and thus $f: \beta X \to M^n$ is also onto. We will now show that any map homotopic to $f$ must also be onto.

Claim 1. The map $f: \beta X \to M^n$ is homotopically onto.

Proof of Claim 1. Suppose that $H: \beta X \times I \to M^n$ be a homotopy from $f$ to $h$. Note that $g: X \to \mathbb{R}^n$ is a lift for the map $f|X: X \to M^n$. Thus there is a unique map $\tilde{H}: X \times I \to \mathbb{R}^n$ such that $c \circ \tilde{H} = H|X \times I$ and $\tilde{H}|X \times \{0\} \equiv g$. Let $\tilde{h} = \tilde{H}|X \times \{1\}$. Then $\tilde{h}$ is a lifting for the map $h|X$. We will show that $h$ is onto by showing that $\tilde{h}$ is onto. Once we have shown this Claim 1 will follow and the proof of Theorem 1.2 will be complete.

Claim 2. The map $\tilde{h}: X \to \mathbb{R}^n$ is onto.

Proof of Claim 2. Suppose that $x \in \mathbb{R}^n$ with $x \not\in \tilde{h}(X)$. Then by Lemma 1.1 there is an open set $U$ in $\mathbb{R}^n$ with $x \in U$ and $\overline{U}$ compact such that if $y \in X$ and $\tilde{h}(y,t) = x$, then $\tilde{h}(\{y\} \times I) \subseteq U$. Let $N$ be an integer such that $N \cdot B^n$ contains $U$. Then let $I_{n-1}$ be the boundary of $N \cdot B^n$ and
$D_N = N \cdot g_N^{-1}(\Sigma^{n-1}) \subset L_N$. Now since $g_N: L_N \to B^n$ was an essential map, $f|D_N = N \cdot g_N|D_N$ cannot be extended to a map of $L_N$ taking values in $\Sigma^{n-1}$. We will now get a contradiction to this. Then Claim 2 will follow.

Claim 3. There must be an extension of $f|D_N$ to $L_N$ taking values in $\Sigma^{n-1}$.

Proof of Claim 3. Now $x$ is in the interior of $N \cdot B^n$. Let $r: R^n \setminus \{x\} \to \Sigma^{n-1}$ be the projection along the rays emanating from $x$. Let $s: L_N \to \Sigma^{n-1}$ be defined by $s = r \circ h$. Then $s$ is defined and continuous since $h(y) \neq x$ for all $y \in X$. Consider the homotopy $\hat{h}|D_N \times I$. Note that if $x \in \hat{h}(D_N \times I)$, then $\hat{h}(y,t) = x$ for some $y \in D_N$ and $t \in I$. But for that $y$, $\hat{h}(y,0) \in \Sigma^{n-1}$ since $\hat{h}(y,0) = g(y)$. This contradicts the choice of $U$ and $N$, since by Lemma 1.1 $\hat{h}(\{y\} \times I) \subset U \subset \text{int } N \cdot B^n$. Thus $x \notin \hat{h}(D_N \times I)$. Thus $r \circ \hat{h}|D_N \times I: D_N \times I \to \Sigma^{n-1}$ is defined and continuous and a homotopy joining the map $f|D_N$ to the map $s|D_N$. However, $s|D_N$ has a continuous extension to all of $L_N$ having values in $\Sigma^{n-1}$, namely $r \circ \hat{h}$. By the Borsuk Extension Theorem $f|D_N$ must also have an extension to $L_N$ with values in $\Sigma^{n-1}$. This proves Claim 3.

Claim 3 is a contradiction of the fact that $f|L_N = N \cdot g_N$ was an essential map. This contradiction shows that $\hat{h}$ must be onto and completes the proof of Claim 2. The fact that $\hat{h}$ is onto shows that $h: \beta X + M^n$ must be onto. Thus $f$ is homotopically onto and the proof of the theorem is complete.

1.3. Theorem. Let $M^n$ be a closed $n$-manifold whose covering space is homeomorphic to $R^n$. Suppose that $X$ is
Locally compact and c-compact such that for every compact set $K \subset X$, there is a compact set $L \subset X-K$ such that $\dim L \geq n$. Then there is a map $f : \beta X - X \to \mathbb{M}^n$ which is homotopically onto.

Proof. This follows from the proof of Theorem 1.2.

Let $f : \beta X + \mathbb{M}^n$ be the map constructed in the proof of Theorem 1.2. We claim that $f|_{\beta X - X}$ is homotopically onto $\mathbb{M}^n$. Suppose not. Then let $h : \beta X - X + \mathbb{M}^n$ be a map homotopic to $f$ which is not onto $\mathbb{M}^n$. We may assume that $h(\beta X - X) \subset \mathbb{M}^n - 0$ where $0$ is an open $n$-ball in $\mathbb{M}^n$ with $\mathbb{M}^n - 0$ a manifold with boundary.

Now $\mathbb{M}^n - 0$ is an ANR. Thus there must be an extension of $h$ to a neighborhood $V$ of $\beta X - X$ in $\beta X$ taking values in $\mathbb{M}^n - 0$. Call this extension $h$. Then $h(V) \subset \mathbb{M}^n - 0$ is also not onto $\mathbb{M}^n$. We can also assume that $h$ and $f|_V$ are homotopic as maps into $\mathbb{M}^n$. Since $V$ is a neighborhood of $\beta X - X$, $X - V = K$ must be compact. Let $N$ be such that for $i \geq N$, $L_i \cap K = \emptyset$. Then $L_i \subset V$ for all $i \geq N$. Let $U$ be an open set containing $K \cup (\bigcup_{i=1}^{N-1} L_i)$ such that $\overline{U}$ is compact with $\bigcup_{i=N}^\infty L_i \subset X - U$. Then let $X' = X - U$. Then one can repeat the proof of Theorem 1.2 to show that $f|_{\beta X'}$ must be homotopically onto $\mathbb{M}^n$. However, $X'$ is a closed subset of $X$ which is normal and thus $cl_{\beta X'} = \beta X'$. This gives us a contradiction since $h|_{cl_{\beta X'}}$ is homotopic to $f|_{cl_{\beta X'}}$ and $h|_{cl_{\beta X'}}$ is not onto. This contradiction shows that $f|_{\beta X - X}$ must be homotopically onto.

This proves Theorem 1.3.

1.4. Corollary. Suppose that $X$ is any normal space which contains a closed discrete set of compact sets $\{L_i\}_{i=1}^\infty$ such that $\dim L_i \geq n$ for all $i$. Let $\mathbb{M}^n$ be a closed $n$-manifold
whose universal covering space is homeomorphic to $\mathbb{R}^n$. Then there is a map $f: \beta X \to M^n$ which is homotopically onto and such that $f|\beta X - X$ is also homotopically onto.

1.5. Remark. There are other extensions of Theorems 1.2 and 1.3 along the lines of Corollary 1.4, but there is not space here to include them.

1.6. Theorem. Let $n \geq 1$ be an integer. Let $X$ be a locally compact and $\sigma$-compact space such that for every compact set $K \subset X$ there is a compact set $L \subset X-K$ such that $\dim L \geq n$. Then $\text{Sd } \beta X \geq n$ and $\text{Sd}(\beta X - X) \geq n$.

Proof. Let $T^n = M^n$ in Theorems 1.2 and 1.3. Then there is a map $f: \beta X \to T^n$ which is homotopically onto and such that $f|\beta X - X$ is also homotopically onto. Now if $\text{Sd } \beta X < n$, then by Theorem 0.4 there would be an $(n-1)$-dimensional compact polyhedron $Q$ and maps $g: \beta X \to Q$ and $r: Q \to T^n$ such that $r \circ g$ is homotopic to $f$. But we can take $r$ to be simplicial and then $\dim r(Q) \leq n-1$. Thus $r \circ g(\beta X)$ would not be all of $T^n$. This is a contradiction. Thus $\text{Sd}(\beta X) \geq n$. Similarly $\text{Sd}(\beta X - X) \geq n$ also.

Actually we will show a much stronger result about the shape dimension of compacta in $\beta X - X$. In Corollary 1.9 we will show that for $X$ Lindelöf and $K$ a compactum contained in $\beta X - X$, $\dim K = \text{Sd } K$. To prove this we will need Theorem 1.8. First we need an easy lemma.

1.7. Lemma. Let $X$ be Lindelöf and $A \subset \beta X - X$ be such that $A$ is Lindelöf and $A$ is closed in $X \cup A$. Then $\text{cl}_{\beta X} A = \beta A$.

Proof. We need to show that every continuous map
f: A → [0,1] extends to a map F: cl_{βX}A → [0,1]. Let
f: A → [0,1]. Then X ∪ A is Lindelöf and A is closed in
X ∪ A. Thus there is an extension of f to X ∪ A. Call this
extension f again. Then f has an extension βf to βX. Then
let F = βf|cl_{βX}A. Clearly F is an extension of our original
map f: A → [0,1].

1.8. Theorem. Let K be a compactum contained in βX−X
where X is a Lindelöf space. Let M^n be as in Theorem 1.2.
Then if dim K ≥ n, then there is a map f: K → M^n which is
homotopically onto.

Proof. Actually we will construct a map f: βX → M^n
such that f|K is homotopically onto. First we will need a
sequence of compact sets L ⊆ K such that {L_i} := i
is a disjoint collection with dim L_i ≥ n for all i.

Claim 1. There is a sequence of compacta L_i ⊆ K such
that {L_i} := i
is a disjoint collection with dim L_i ≥ n for
all i.

Proof of Claim 1. Suppose that for each x ∈ K, there is
a set U_x open in K containing x such that dim U_x < n. Then
one could cover K by a finite number of such open sets,
{U_1, ..., U_K} with dim U_1 < n. But then by the Sum Theorem for
dimension (Theorem 0.2), dim K < n, a contradiction. Thus
there must be an x_1 ∈ K such that for every open set U con­
taining x_1, dim U > n. Now there must be a set U_1 open in
K and containing x_1 such that dim K − U_1 ≥ n (by [4, Corollary
7, p. 80]). Then let V_1 be a set containing x_1 and open in
K such that V_1 ⊆ U_1. Then dim V_1 ≥ n and dim(K − U_1) ≥ n with
V_1 ∩ (K − U_1) = Ø. Now let x_2 ∈ K − U_1 be such that every set U
containing $x_2$ and open in $K-U_1$ has the property that $\dim U \geq n$. Then let $U_2$ be a set open in $K-U_1$ containing $x_2$ such that $\dim(K-(U_1 \cup U_2)) \geq n$. Then let $V_2$ contain $x_2$ with $V_2$ open in $K-U_1$ such that $V_2 \subseteq U_2$. Then $V_1 \cap V_2 = \emptyset$ and $\dim V_2 \geq n$ and $\dim(K-(U_1 \cup U_2)) \geq n$. Continuing this process one gets a sequence of sets $\{V_i\}_{i=1}^\infty$ such that $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\dim V_i \geq n$ for all $i$. Then $L_i = V_i$ are the required compacta in $K$. This proves Claim 1.

We now resume the proof of Theorem 1.8. Let $B^n$ be the closed unit ball in $\mathbb{R}^n$ and let $g_i: L_i \to B^n$ be an essential map. Let $g: \bigcup_{i=1}^\infty L_i \to \mathbb{R}^n$ be defined by $g(x) = k \cdot g_k(x)$ for $x \in L_k$. Now $\bigcup_{i=1}^\infty L_i$ is Lindelöf. Also $K \cap (X \cup (\bigcup_{i=1}^\infty L_i)) = \bigcup_{i=1}^\infty L_i$ and thus $\bigcup_{i=1}^\infty L_i$ is closed in $X \cup (\bigcup_{i=1}^\infty L_i)$. Thus there is an extension $g: X \cup (\bigcup_{i=1}^\infty L_i) \to \mathbb{R}^n$. Let $c: \mathbb{R}^n + M^n$ be the covering map and $f: \beta X + M^n$ be the Čech extension of the map $c \circ g: X + M^n$. Then we claim that $f|K$ is homotopically onto.

Claim 2. The map $f|K$ is homotopically onto $M^n$.

Proof of Claim 2. Let $D = \text{cl}_{\beta X}(\bigcup_{i=1}^\infty L_i)$. Then $D \subseteq K$ and actually $f|D$ is homotopically onto. The proof proceeds exactly as in the proof of Theorem 1.2. The only observation that needs to be made is that $\text{cl}_{\beta X}(\bigcup_{i=1}^\infty L_i)$ is equivalent to $\beta(\bigcup_{i=1}^\infty L_i)$ so that we may make use of Lemma 1.1. However, Lemma 1.7 shows that $\text{cl}_{\beta X}(\bigcup_{i=1}^\infty L_i) = \beta(\bigcup_{i=1}^\infty L_i)$. The proof of Claim 2 is now clear. Claim 2 completes the proof of Theorem 1.8.

1.9. Corollary. Let $X$ be a Lindelöf space and let $K$ be a compactum contained in $\beta X - X$. Then $\dim K = \text{Sd} K$.

Proof. This is clear if $\dim K = 0$ or $-1$. If
dim $K \geq n \geq 1$, then by Theorem 1.8 there is a continuous map $f(K) = T^n$ which is homotopically onto. Thus $Sd K \geq n$ also. Thus $Sd K \geq \dim K$. However, $Sd K \leq \dim K$ by definition. Thus $Sd K = \dim K$.

1.10. Example. The above results imply that the shape dimension of $\beta R^n$ and $\beta R^n - R^n$ is $n$ and thus that the dimension of these spaces is $n$ also. By the results of Calder and Siegel [1] we have that for $k \geq 2$, $H^k(\beta R^n) = 0 = H^k(R^n)$. Thus for each $n \geq 2$ there is a continuum $X_n = \beta R^n$ such that $Sd X_n = n$ with $H^k(X_n) = 0$ for all $k \geq 2$. In the next section we will show that this implies the existence of a metric continuum $X_n$ having these properties. This will provide a counterexample to a question raised by Nowak in [11].

1.11. Example. Let $L$ be the long line and $X = L \times I^n$. Then $\beta (X \times I^n) = (L \cup \{w_1\}) \times I^n$. The space $\beta X$ has trivial shape as does $\beta X - X = \{w_1\} \times I^n$. Thus $Sd \beta X = 0 = Sd(\beta X - X)$. However, $\dim \beta X = \dim(\beta X - X) = n$. Of course $X$ is not $\sigma$-compact or Lindelöf. This shows that without some assumption on $X$ one cannot expect any of the theorems in this section to hold.

2. Metric Continua and Polyhedra

In this section we apply the results of the first section to show the existence of metric continua and polyhedra which have surprising properties. One of the examples answers a question of Nowak [11]. We proceed with the examples.

2.1. Theorem. For each $n \geq 2$ there is a finite
polyhedron $P_n$ of dimension $n$ such that there is a map
$f: P_n \rightarrow T^n$ with $f$ homotopically onto and with $f^*: H^k(T^n) \rightarrow H^k(P_n)$ the zero-homomorphism for all $k \geq 2$.

Proof. Let $X = \beta R^n$. Then there is a map $g: X \rightarrow T^n$
which is homotopically onto. Now $H^k(\beta R^n) = 0$ for all $k \geq 2$
by [1]. Thus $g^*: H^k(T^n) \rightarrow H^k(X)$ is the zero-homomorphism.

Now $\dim \beta R^n = n$ and thus there is a cofinal set of finite
open covers of $X$ having order at most $n+1$. Let \{${U_{\alpha}}$ $\alpha \in A$\} be
this set. Now $H^*(X) = \lim_{\rightarrow} H^*(N(U_{\alpha}))$ where $N(U_{\alpha})$ is the nerve
of $U_{\alpha}$ and the bonding homomorphisms are induced by the pro-
jection maps $\pi_{\alpha\beta}: N(U_{\beta}) \rightarrow N(U_{\alpha})$ for $U_{\beta}$ a refinement of $U_{\alpha}$.

Now for a cofinal collection of the $\alpha$'s, there are maps
$g_{\alpha}: X \rightarrow N(U_{\alpha})$ and $r_{\alpha}: N(U_{\alpha}) \rightarrow T^n$ such that $r_{\alpha} \circ g_{\alpha}$ is homotopic
to $g$ where $g_{\alpha}: X \rightarrow N(U_{\alpha})$ is a barycentric map for the cover
$U_{\alpha}$. Note that $r_{\alpha}: N(U_{\alpha}) \rightarrow T^n$ must be homotopically onto
since $g$ is homotopically onto. Now $H^k(X) = \lim_{\rightarrow} H^k(N(U_{\alpha}))$
and thus $g^* = \lim_{\rightarrow} r_{\alpha}^*: H^k(T^n) \rightarrow H^k(N(U_{\alpha}))$. Since $H^k(T^n)$ is
finitely generated for all $k$ and $g^* = 0$ for all $k \geq 2$, there
must be a $\beta$ such that $r_{\beta}^*: H^k(T^n) \rightarrow H^k(N(U_{\alpha}))$ is the zero-
homomorphism for all $\alpha \geq \beta$ and all $k \geq 2$. Then let $P_n = N(U_{\beta})$ and $f = r_{\alpha}$. This proves Theorem 2.1.

The above example has the property that for any map
$g: T^n \rightarrow S^n$, $g \circ f: P_n \rightarrow S^n$ is null-homotopic. This is by
Theorem 0.5 since $\dim P_n = n$. One would tend to think that
the map $f: P_n \rightarrow T^n$ could not be degree zero and still be
homotopically onto. However, this is in fact the case and
the result is somewhat surprising.

2.2. Theorem. Let $n \geq 2$. Then there is a metric
continuum $X_n$ such that $\dim X_n = Sd X_n = n$ and such that $H^k(X_n) = 0$ for all $k \geq 2$.

Proof. Let $Y_n = \beta R^n$. Let $g: Y_n \to T^n$ be a map which is homotopically onto. Then let $\{U_\alpha: \alpha \in A\}$ be a cofinal set of open covers of $Y_n$ such that the order of $U_\alpha$ is at most $n + 1$ for all $\alpha \in A$. There must be a $\beta \in A$ such that there are maps $r_\beta: N(U_\beta) \to T^n$ and a barycentric map $g_\beta: Y_n \to N(U_\beta)$ such that $r_\beta \circ g_\beta$ is homotopic to $g: Y_n \to T^n$. Then for all $\alpha \geq \beta$ the maps $g_\alpha: Y_n \to N(U_\alpha)$ and $\pi_\beta \alpha: N(U_\alpha) \to N(U_\beta)$ have the property that $r_\beta \circ \pi_\beta \alpha \circ g_\alpha$ is homotopic to $g$. Thus $r_\beta \circ \pi_\beta \alpha: N(U_\alpha) \to T^n$ must be homotopically onto since $g$ is homotopically onto. Now $H^*(Y_n) = \lim_{\alpha} H^*(N(U_\alpha)): \alpha \geq \beta$. By an inductive process one can construct a sequence $\beta = a_1 < a_2 < \cdots$ in $A$ such that $\lim_{\alpha} H^k(N(U_\alpha)) = 0$ for all $k \geq 2$. One uses the fact that $H^k(Y_n) = 0$ for all $k \geq 2$ together with the fact that $H^k(N(U_\alpha))$ is finitely generated for all $i$. Now let $\pi_i: N(U_{a_{i+1}}) \to N(U_a)$ be a projection map for each $i$. Then let $X_n = \lim_{\alpha} N(U_a)$; $\pi_i$. Then $X_n$ will be a metric continuum. The reason is that each of the $N(U_a)'s$ will be connected since $\beta R^n$ is connected. Also $\dim X_n \leq n$ since $\dim N(U_a) \leq n$ for all $i$. Also $H^k(X_n) = \lim_{\alpha} H^k(N(U_a)); \pi_i = 0$ for all $k \geq 2$.

Now suppose that $h_i: X_n \to N(U_a)$ are the maps making $X_n$ the inverse limit of the inverse system $\{N(U_a)\}$. Now consider $g_\beta \circ h_1: X_n \to T^n$. We claim that $g_\beta \circ h_1$ is homotopically onto.

Claim. The map $g_\beta \circ h_1$ is homotopically onto.

Proof of Claim. Suppose not. Then there must be a $k$ such that $g_\beta \circ \pi_1 \circ \cdots \circ \pi_k: N(U_k) \to T^n$ is not homotopically onto. However, this implies that $g_\beta \circ \pi_\beta \alpha_{k+1}$ is not
homotopically onto since \( \pi_{1} \cdots \pi_{k} \) is a projection map \( \pi_{\beta \alpha k+1} \) from \( N(\alpha \beta k+1) \) to \( N(\beta \alpha k+1) \). This is a contradiction since we have already remarked that \( g_{\beta} \circ \pi_{\beta \alpha} \) is homotopically onto for all \( \alpha > \beta \). Thus \( g_{\beta} \circ h_{1} \) must be homotopically onto as asserted in the claim.

Now by the Claim, \( g_{\beta} \circ h_{1} : X_{n} \to T^{n} \) is homotopically onto. Thus \( Sd(X_{n}) = Fd(X_{n}) > n \). Since \( \dim X \leq n \), we must have \( \dim X_{n} = Sd X_{n} = n \). We have already shown that \( H^{k}(X_{n}) = 0 \) for all \( k \geq 2 \). Thus \( X_{n} \) has the desired properties and Theorem 2.2 is proved.

2.3. Remark. Theorem 2.2 solves Problem 6.7 in [11]. J. Hollingsworth also has an example which solves this problem.

2.4. Corollary. Let \( n \geq 2 \). Then there exists a metric continuum \( X_{n} \) such that \( Sd X_{n} = \dim X_{n} = n \) with \( Sd(\Sigma X_{n}) = 2 \) where \( \Sigma X_{n} \) is the suspension of \( X_{n} \).

Proof. Let \( X_{n} \) be the example in Theorem 2.2. Then \( Sd(\Sigma X_{n}) = 2 \) by [11, Theorem 4.4].

3. Maps onto \( S^{n} \vee S^{1} \)

In this section we show that there are maps of \( \beta X \) which are homotopically onto other finite polyhedra \( P \) besides manifold \( K(\pi,1)'s \). In particular we show that this is true for \( P = S^{n} \vee S^{1} \). First we make the following observations about maps onto wedges of manifold \( K(\pi,1)'s \). The proofs are only indicated since they are a straight forward modification of the proofs in section one.

3.1. Theorem. Let \( n \geq 1 \). Let \( n_{1}, \cdots, n_{k} \) be integers
such that $n_i \leq n$ for all $i$ and suppose that $M_i$ is a closed $n_i$-manifold whose covering space is $\mathbb{R}^{n_i}$ for $i = 1, \cdots, k$. Suppose that $X$ is a locally compact $\sigma$-compact space such that for every compact set $K \subset X$ there is a compact set $L \subset X - K$ such that $\dim L \geq n$. Then there is a map $f : \beta X + \bigvee_{i=1}^{k} M_i$ which is homotopically onto.

3.2. Theorem. Let $n \geq 1$. Let $n_1, \cdots, n_k$ be integers such that $n_i \leq n$ for all $i$ and suppose that $M_i$ is a closed $n_i$-manifold whose covering space is $\mathbb{R}^{n_i}$ for $i = 1, \cdots, k$. Suppose that $X$ is Lindelöf and that $K$ is a compactum contained in $\beta X - X$ with $\dim K \geq n$. Then there is a map $f : K \to \bigvee_{i=1}^{k} M_i$ which is homotopically onto.

Indication of Proof of 3.1. Let $\{L_i\}_{i=1}^{\infty}$ be disjoint compact subsets of $X$ with $\dim L_i \geq n$ as in the proof of Theorem 1.2. Then break this up into $k$ infinite collections: $\{L_i(1)\}_{i=1}^{\infty}, \cdots, \{L_i(k)\}_{i=1}^{\infty}$. Then let $g_i(j) : L_i(j) + B^{n_j}$ be an essential mapping where $B^{n_j}$ is the closed unit ball in $\mathbb{R}^{n_j}$. Then define a map $g : X + \bigvee_{j=1}^{k} R^{n_j}$ by

$$g|_{L_i(j)} = i \cdot g_i(j) : L_i(j) + R^{n_j}$$

This defines $g$ on $\bigcup_{j=1}^{k} \bigvee_{i=1}^{\infty} L_i(j)$ and then extend in any fashion to all of $X$. Then let $c_j : R^{n_j} + M_j$ be the covering map for $j = 1, \cdots, k$ and let $\bigvee_{j=1}^{k} c_j : \bigvee_{j=1}^{k} R^{n_j} + \bigvee_{j=1}^{k} M_j$ be the wedge of the $c_j$'s. Then let $f : \beta X + \bigvee_{j=1}^{k} M_j$ be the Čech extension of the map

$$(\bigvee_{j=1}^{k} c_j) \circ g : X + \bigvee_{j=1}^{k} M_j.$$ 

Then one can show that $f$ is homotopically onto in a manner similar to the proof of Theorem 1.2.

The proof of Theorem 3.2 is a similar modification of
the proof of Theorem 1.8. We now proceed to maps which are homotopically onto $S^n \vee S^1$. The following proposition follows from Theorem 10.3 of [16]. We include a simple proof for completeness.

3.3. Proposition. Let $X$ be a paracompact space of dimension $n$. Then there is a closed set $C \subset X$ and a map $f: (X,C) \to (B^n,S^{n-1})$ such that $f$ is not null-homotopic as a map of pairs and if $e: (B^n,S^{n-1}) \to (S^n,p)$ is the map which takes all of $S^{n-1}$ to $p$, then $e \circ f: (X,C) \to (S^n,p)$ is also not null-homotopic as a map of pairs.

Proof. Let $C$ be a closed subset of $X$ so chosen that there is a map $f: C \to S^{n-1}$ which cannot be extended to all of $X$ with values in $S^{n-1}$. Let $i: C \to X$ be the inclusion map. Let $f: X \to B^n$ be an extension of our original map $f: C \to S^{n-1}$. Let $e: B^n \to S^n$ be the quotient map taking $S^{n-1}$ to $p \in S^n$. Then the following diagram commutes and $e^*$ is an isomorphism.

\[
\begin{array}{ccc}
H^n(S^n,p) & \xrightarrow{e^*} & H^n(B^n,S^{n-1}) \\
\downarrow{\delta_1} & & \downarrow{\delta_2} \\
H^{n-1}(S^{n-1}) & \xrightarrow{f^*} & H^{n-1}(X) \\
\uparrow{i^*} & & \\
H^{n-1}(X)
\end{array}
\]

Now by the Hopf Extension Theorem (Theorem 0.3), there must be an $h \in H^{n-1}(S^{n-1})$ with $f^*(h) \not\in i^*H^{n-1}(X)$ or $f$ would have an extension to $X$ with values in $S^{n-1}$. Thus $\delta_2 f^*(h) \neq 0$ in $H^n(X,C)$. Thus $f^* \delta_1(h) \neq 0$ and $f^*: H^n(B^n,S^{n-1}) \to H^n(X,C)$ is not the zero homomorphism. Thus $f: (X,C) \to (B^n,S^{n-1})$ cannot be null-homotopic. Also, since $e^*$ is an isomorphism,
3.4. Theorem. Let $n \geq 2$. Suppose that $X$ is a locally compact $\sigma$-compact space such that for every compact subset $K$ of $X$ there is a compact set $L \subset X$ with $\dim L = n$ and with $L \cap K = \emptyset$. Then there is a map $f$ of $\beta X$ onto $S^n \vee S^1$ which is homotopically onto.

Proof. Let $\{L_i\}_{i=1}^{\infty}$ be a sequence of compact subsets of $X$ having the following properties: (1) $\dim L_i = n$, (2) $L_i \cap L_j = \emptyset$ for $i \neq j$, and (3) if $K$ is a compact subset of $X$, then there is an $i$ such that $L_i \cap K = \emptyset$. The construction of such a sequence of $L_i$'s has been carried out in the proof of Theorem 1.2. Now let $C_i$ be a closed subset of $L_i$ and $f_i: (L_i,C_i) \rightarrow (B^n,S^{n-1})$ be a map of pairs which is not null-homotopic such that $ef_i$ is also not null-homotopic where $e: (B^n,S^{n-1}) \rightarrow (S^n,p)$. Such maps $f_i$ exist by Proposition 3.3.

Now let $A$ be the universal covering space for $S^n \vee S^1$ and $c: A \rightarrow S^n \vee S^1$ the covering map. We can think of $A$ as the countable union of $n$-spheres attached to the real line $R$ at the integer points. Now let $B$ be the countable union of closed unit $n$-balls, $B^n$, attached at the integer points of $R$. 

![Diagram of B^n and B_0 and B_1]

502 Keesling

eof: $(X,C) \rightarrow (S^n,p)$ cannot be null-homotopic either.
Now define a map \( g: X \to B \) by \( g|_{L_i} = f_i \) for each \( i \).
Then let \( g \) be any extension to all of \( X \). Such an extension exists since \( B \) is an absolute extensor. Let \( B^n_i \) be the \( n \)-ball attached at the integer point \( i \) in \( B \). Let \( S^n_i \) be the \( n \)-sphere attached at the integer point \( i \) in \( A \). Let \( i > 1 \) and let \( h_i: B_i \to A \) be defined in the following manner. Let \( D_i \) be a closed collar for the boundary \( S^{n-1}_i \) of \( B^n_i \) and let \( E^{n-1}_i \) be the interior component of the boundary of \( D_i \). Then let \( h_i \) take \( E^{n-1}_i \) to the point \( i \in A \). Then let \( h_i(S^{n-1}_i) \) map to the point \( 2i \in A \) and let \( h_i|_{D_i} \) map onto the arc \( [i, 2i] \subset R \subset A \). Then let \( h_i \) map the interior of \( E^{n-1}_i \) homeomorphically onto \( S^n_i - \{i\} \).

Then define \( k_i: L_i \to A \) by \( k_i = h_i \circ f_i = h_i \circ g \). Then we make the following observation.

**Claim 1.** If \( m_i: L_i \to A \) is homotopic to \( k_i \) by a homotopy \( H \) such that at each stage \( t \) of the homotopy \( H(C_i, t) \subset A - S^n_i \), then \( m_i(L^n_i) \) contains \( S^n_i \).
**Proof of Claim 1.** We can convert the homotopy $H$ to one which maps to $S^n_1$ by projecting all of $A - S^n_1$ to the point $i$. Let $H'$ be this homotopy and let $m'_1$ and $k'_1$ be the corresponding ends of the homotopy. Then the homotopy is actually a homotopy of pairs $H': (L_1, C_1) \times I \to (S^n_1, i)$ from $k'_1$ to $m'_1$. Now $k'_1$ is homotopic as a map of pairs to $e \circ f'_1$ (or to the map $e \circ f'_1$ followed by an orientation reversal of $S^n_1$). Thus the map $m'_1$ is also homotopic to $f'_1$ as a map of pairs. This implies that $m'_1(L_1) = S^n_1$ and thus that $m_1(L_1)$ contains $S^n_1$. This proves Claim 1.

We now proceed with the proof of Theorem 3.4. Let $h: B \to A$ be any map such that $h_iB_i = h_i$ for $i = 1, 2, \ldots$. We have already defined a map $g: X \to B$ and the covering map $c: A \to S^n_1$. Then let $q: X \to S^n_1$ be defined by $q = c \circ h \circ g$. Then let $f = \beta q: \beta X \to S^n_1$. We will now show that this $f$ is homotopically onto.

**Claim 2.** If $p: \beta X \to S^n_1$ is homotopic to $f$, then $S^n$ is in the image of $p$.

**Proof of Claim 2.** Suppose that $p$ is homotopic to $f$. Then $f|X$ has a lift to $A$, namely the map $h \circ g: X \to A$. Call this map $\tilde{f}$. Let $H$ be the homotopy from $f$ to $p$ and let $\tilde{H}$ be the lift of this homotopy starting at $\tilde{f}$ and ending at $\tilde{p}$. Consider the point $0 \in R \subset A$. By Lemma 1.1 there is an open set $U$ in $A$ with $\overline{U}$ compact with $0 \in U$ such that if $x \in X$ and $h: \{x\} \times I \to A$ is any lift of $H|\{x\} \times I$, then if $0 \in h(\{x\} \times I)$, then $h(\{x\} \times I) \subset U$. Now let $N$ be an integer such that if $r$ is the projection of $A$ onto $R$ taking $S^n_1$ to $i$ for each $i$, then the projection of $U$ is contained in $(-N,N)$. Then for
any \( x \in X \), if the homotopy \( \tilde{H} \) has the property that \( \tilde{H}({\{x\} \times I}) \) contains \( i \in R \subset A \), then \( r \circ \tilde{H}({\{x\} \times I}) \subset (i-N, i+N) \). Now let \( M > N \). Then \( h \circ g|_M \) maps onto \( S^0 \cup \{M, 2M\} \subset A \) with \( h \circ g(C_M) = \{2M\} \). Now \( \tilde{H}|_M \times I \) must have the property that \( r \circ \tilde{H}(C_M \times I) \) does not contain the point \( M \) for all \( t \in I \) since \( 2M \in \tilde{H}(C_M \times I) \) and \( 2M-M = M > N \). Thus \( \tilde{H}(C_M \times I) \subset A - S^0_M \). Thus the map \( \tilde{p}|_M \) maps onto \( S^0_M \) by Claim 1 (since \( k \) in Claim 1 is just \( h \circ g|_M = f|_M \)). Thus \( \tilde{p}|_M \) maps onto \( S^n \). Thus \( p \) contains \( S^n \) in its image. This proves Claim 2.

Claim 3. If \( p: SX \rightarrow S^n \cup S^1 \) is homotopic to \( f \), then \( p \) contains \( S^1 \) in its image.

Proof of Claim 3. Let \( H \) be the homotopy joining \( f \) to \( p \). Let \( \tilde{f} = h \circ g : X \rightarrow A \) and let \( \tilde{H} : X \times I \rightarrow A \) be the lift of \( H|X \times I \) joining \( \tilde{f} \) to \( \tilde{p} \) as in Claim 2. Let \( N \) be a positive integer such that if \( x \in X \) and \( 0 \in h({\{x\} \times I}) \) for some lift of \( H|\{x\} \times I \), then \( r \circ h({\{x\} \times I}) \subset (-N, N) \) where \( r : A \rightarrow R \) is the retraction of \( A \) to \( R \) which takes each \( S^0_i \) to the point \( i \).

Let \( M \) be a positive integer with \( M > 2N+1 \). Then noting that \( \tilde{f}(C_M) = \{2M\} \subset R \) and \( \tilde{f}(y) = M \) for some \( y \in L_M \) we must have that for all \( x \in C_M \), \( r \circ \tilde{p}(x) > 2M-N \) and for \( y \), \( r \circ \tilde{p}(y) < M+N \). Thus for all \( x \in C_M \), \( r \circ \tilde{p}(x) - r \circ \tilde{p}(y) > 2M - N - (M+N) = M - 2N > 1 \). Now by the proof of Claim 2, \( \tilde{p}(L_M) = S^0_M \). Now we want to claim that \( \tilde{p}(L_M) \) contains all of the interval \( [M+N, 2M-N] \). If it does, then this interval maps onto \( S^1 \) by \( c : A \rightarrow S^n \cup S^1 \). Thus \( p \) will contain \( S^1 \) in its image and Claim 3 will be proved. Thus Claim 3 will follow from Claim 4.

Claim 4. The map \( \tilde{p}|_M \) contains \( [M+N, 2M-N] \) in its
image.

Proof of Claim 4. Suppose not and suppose that \( z \in [M + N, 2M - N] \) with \( z \notin \tilde{p}(L_{M}) \). We may assume \( z \) is not an integer point since the points in \([M + N, 2M - N]\) not in the image of \( \tilde{p}|_{L_{M}} \) is an open set. Then let \( L_{M} = C \cup D \) where \( C = (r \circ \tilde{p}|_{L_{M}})^{-1}(\partial, z) \) and \( D = (r \circ \tilde{p}|_{L_{M}})^{-1}(z, +\infty) \). This is a separation of \( L_{M} \) with \( C_{M} \subset D \). Now since \( \tilde{p}(L_{M}) \) contains \( S_{M}^{n} \), it must be that \( \tilde{p}(C) \supseteq S_{M}^{n} \) since \( \tilde{p}(D) \cap S_{M}^{n} = \emptyset \). However, \( \tilde{p}|C \) is homotopic to \( \tilde{f}|C \) in \( A \) and \( \tilde{f}|C : C \rightarrow A \) factors through \( B_{M}^{n} \). Thus \( \tilde{f}|C \) is null-homotopic and \( \tilde{p}|C \) is null-homotopic. Thus \( \tilde{p}|L_{M} \) is homotopic to a map \( v \) by a homotopy \( H' \) with the property that \( v|C \) is constant and \( H'|D \times I \equiv \tilde{p}|D \). But then \( \tilde{f}|L_{M} \) is homotopic to the map \( v \) by a homotopy \( H'' \) which joins the homotopies \( H \) and \( H' \). The homotopy \( H'' \) has the property that \( H''(C_{M} \times I) \subset A-S_{M}^{n} \) and the map \( v \) does not contain \( S_{M}^{n} \) in its image. This contradicts Claim 1. This contradiction shows that \( \tilde{p}(L_{M}) \) contains all of \([M + N, 2M - N]\) and Claim 4 is proved.

3.5. Theorem. Let \( X \) be locally compact and \( \sigma \)-compact and \( n \geq 2 \) be an integer. Suppose that \( X \) has the property that for every compact set \( K \subset X \) there is a compact set \( L \subset X-K \) with \( \dim L = n \). Then there is a map \( f: \beta X-X \rightarrow S_{n}^{v}S^{1} \) which is homotopically onto.

Proof. The proof is a modification of the proof of Theorem 1.3.

3.6. Theorem. Let \( K \) be a compactum contained in \( \beta X-X \) where \( X \) is a Lindelöf space. Let \( n \geq 2 \). Then if \( \dim K = n \), then there is a map \( f: K \rightarrow S_{n}^{v}S^{1} \) which is homotopically onto.
Proof. The proof is similar to the proof of Theorem 1.8.

3.7. Question. Is it true that if $X$ is any paracompact space with $\dim X \geq n$, then there is a closed set $C \subset X$ and a map $f: (X,C) \rightarrow (B^n,S^{n-1})$ which is not null-homotopic such that if $e: (B^n,S^{n-1}) \rightarrow (S^n,p)$ collapses $S^{n-1}$ to $p$, then $e \circ f: (X,C) \rightarrow (S^n,p)$ is also not null-homotopic?

3.8. Example. One problem in answering Question 3.7 is that we do not have a nice relationship between maps into $n$-spheres and cohomology for infinite-dimensional spaces. For instance for each odd prime $p$ Kahn [6] has given an example of an infinite-dimensional metric continuum $X_p$ such that $H^k(X_p) = 0$ for all $k > 0$, but with $X_p$ having essential maps onto $S^{3+i(2p-2)}$ for all $i > 0$.

References

[8] ______, Decompositions of the Stone-Čech compactification...
which are shape equivalences, Pac. J. Math. (to appear).

[9] _______ and R. B. Sher, Shape properties of the Stone-
Čech compactification, Gen. Top. and Its Appl. (to ap-
pear).

1970.


and Its Appl. 3 (1973), 265-282.

1966.

Verlag, New York, 1974.


[16] C. H. Dowker, Mapping theorems for noncompact spaces,

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