REMARKS ON lambda-COLLECTIONWISE HAUSDORFF SPACES

by

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The purpose of this note is to answer questions raised by Fleissner in [F]. Explicitly, our results are

Theorem 1. Let $\Sigma$ be the statement "there is a locally countable, locally compact, normal Moore space which is $\leq_{\omega_1}$-collectionwise Hausdorff but not $\leq_{\omega_2}$-collectionwise Hausdorff." $\Sigma$ is consistent with ZFC (the usual axioms for set theory). Moreover, both $\Sigma + \neg CH$ and $\Sigma + CH$ are consistent with ZFC.

Theorem 2. Let $M$ be a model of set theory obtained by using Levy forcing to collapse a weakly compact cardinal to $\omega_2$. In $M$, let $X$ be a locally countable space. Then $X$ is $\leq_{\omega_2}$-collectionwise Hausdorff if $X$ is $\leq_{\omega_2}$-collectionwise Hausdorff.

There are variations on Theorem 2. We may replace "locally countable" with "first countable and locally of cardinality $\leq_{\omega_1}$." Also, if we collapse a supercompact cardinal (rather than a merely weakly compact cardinal), we may strengthen the conclusion to $X$ is collectionwise Hausdorff.

A subset $Y$ of a topological space $X$ is called closed, discrete if every point of $X$ has a neighborhood containing

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at most one point of $Y$. A closed discrete set $Y = \{y_i : i \in I\}$ can be screened if there is a family of disjoint open sets

\{U_i : i \in I\} such that $U_i \cap Y = \{y_i\}$. A space $X$ is called

\emph{collectionwise Hausdorff} if every closed discrete subset of

$X$ can be screened. $X$ is $<\lambda$-\emph{collectionwise Hausdorff} if every
closed discrete subset of cardinality $<\lambda$ can be screened;

$\leq\lambda$-\emph{collectionwise Hausdorff} is defined similarly.

On the situation of Theorem 1 for $\mathcal{E} + \text{GCH}$, see [S'].

1. Proof of Theorem 1

For concreteness, let us start with a model of $V = L$.

Then by Jensen's work [J], there is a subset $E$ of $\omega_2$ such

that

a) $\alpha \in E$ implies $\text{cf} \alpha = \omega$

b) $E$ is stationary in $\omega_2$

c) $E \cap \delta$ is not stationary in $\delta$ for any $\delta < \omega_2$.

For each $\alpha \in E$, choose $\eta_\alpha : \omega + \alpha$ to be strictly increas­
ing with range $\eta_\alpha$ cofinal in $\alpha$. Set $I = \{\eta_\alpha | m: \alpha \in E m \in \omega\}$.

The point set of our space is $E \cup I$. (Note that $E$ and $I$ are

disjoint). Points of $I$ are isolated; the nth neighborhood

of $\alpha \in E$, $B(\alpha,n)$, is $\{\alpha\} \cup \{\eta_\alpha|m: m > n\}$. Where we consider

$E$ as a subset of $X$ we will call it $Y$; it is easy to check

that $Y$ is closed, discrete.

From b) and the Pressing Down Lemma it quickly follows

that $Y$ can not be screened, and so $X$ is not $\leq \omega_2$-\emph{collectionwise

Hausdorff}. It is straightforward to prove by induction on

$\rho < \omega_2$ that $\{\alpha \in Y : \alpha < \rho\}$ can be screened. Thus $X$ is

$\leq \omega_1$-\emph{collectionwise Hausdorff}. Clearly $X$ is locally countable, 

locally compact Moore space. This space has been described
in [F'].

In L X is not normal, so our plan is to extend to a model in which X is normal. Of course, we must check that a), b), and c) are preserved. First all a) and c) assert is that certain sets exist, and so are preserved by any extension. Further we want $\omega_2$ in the ground model to remain $\omega_2$ in the extension so that c) has its intended meaning. This will happen because our two extensions are ccc, and $\omega_2$-cc and $\omega$-Baire, respectively. Finally we note that b) is preserved by an $\omega_2$-cc extension. To see this, note that the $\alpha$th element of $C^\alpha$, a club set in the extension, is contained in a set in the ground model of cardinality $\omega_1$. Using this fact, we can find point that are limits of elements of $C^\alpha$ whatever $C^\alpha$ is. Thus we can find for every club set $C^\alpha$ in the extension a club set $C$ in the ground model satisfying $C \subseteq C^\alpha$. We conclude that a set stationary in $\omega_2$ in the ground model remains stationary in $\omega_2$ in the extension. Our extensions will preserve a), b), and c), and X will remain $\leq_{\omega_1}$-collectionwise Hausdorff and not $\leq_{\omega_2}$-collectionwise Hausdorff.

The first extension is the Solovay-Tennenbaum extension forcing Martin's Axiom and $c > \omega_2$, a ccc extension [ST]. We aim at showing that X is normal in this model. It is sufficient to show that disjoint subsets H and K of Y can be separated by disjoint open sets. Define P to be the set of pairs $(u,v)$ satisfying

d) $u \cap v = \emptyset$
e) $u$ (respectively, $v$) is the union of finitely many basic open sets $B(\alpha,n)$ with $\alpha \in H$ (respectively, $\alpha \in K$).
Define \((u,v) \leq (u',v')\) if \(u \supseteq u'\) and \(v \supseteq v'\). That this partial order has ccc follows quickly from the Delta System Lemma. Since \(c > \omega_2 = \text{card } H \cup K\), by Martin's Axiom we can define disjoint open sets \(U\) and \(V\) separating \(H\) and \(K\).

We have shown that \(\Sigma + \text{not CH}\) is consistent. To show that \(\Sigma + \text{CH}\) is consistent, we need a Martin's Axiom-like extension which adds no subsets of \(\omega\). Analogues of Martin's Axiom have been shown consistent and investigated \([T], [S]\), but they are not applicable in this situation because we need a notion of forcing which is not countably closed. Our plan is to make \(X\) normal by an extension which is not countably closed, but is "sufficiently" countably closed.

Let \(H\) and \(K\) be disjoint subsets of \(Y\). We will define a partial order \(P(H,K)\) of pairs \((u,v)\) parallel to the order \(P\) used above with Martin's Axiom. Requirement d) remains the same, but in order to make \(P(H,K)\) sufficiently countably closed, we change e) to

\[e'\) \(u\) (respectively, \(v\)) is the union of countably many basic open sets \(B(\alpha,n)\) where \(\alpha \in H\) (respectively, \(\alpha \in K\)).

Now a new problem arises. It can happen that there is a \(y \in \text{closure } u \cap K\). If this occurs, \((u,v)\) cannot be extended to \((u',v')\) with \(y \in v\), and so the generic filter need not define a separation of \(H\) and \(K\). To prevent this we add, defining \(s((u,v))\) to be \([\text{closure } (u \cup v)] \cap Y\),

\[e''\) \(s((u,v)) \subseteq u \cup v\).

We now define \(P(H,K)\) to be the set of pairs \((u,v)\) satisfying d), e') and e''). \(P(H,K)\) is not countably closed, for the "union" of a countable sequence of elements of \(P(H,K)\) will
satisfy d) and e'), but might not satisfy e'). A sufficient condition for \((u,v)\) to satisfy e" is that \(s((u,v))\) be closed in \(E\) (with the topology inherited from \(\omega_2\) with the order topology). Lemma 1, below will give us a way to insure that certain countable sequences of elements of \(P(H,K)\) will have an infimum. (In our application, the \(A_\alpha's\) will be \(s((u,v))'s\).

Call a well ordered sequence of sets, \(\{A_\alpha: \alpha < \rho\}\), continuous and increasing if

f) \(\alpha < \beta\) implies \(A_\alpha \subseteq A_\beta\)

g) \(\delta\) a limit ordinal implies \(A_\delta = \bigcup\{A_\alpha: \alpha < \delta\}\)

Lemma 1. Suppose that \(E\) satisfies a), b), c); \(\nu\) is an ordinal less than \(\omega_2\); and \(\{A_\alpha: \alpha < \omega_1\}\) is a continuous increasing sequence of countable sets with \(\bigcup\{A_\alpha: \alpha < \omega_1\} = E \cap \nu\). Give \(E \cap \nu\) the topology inherited from \(\omega_2\) with the order topology. Then

\[\{\alpha: A_\alpha \text{ is closed in } E \cap \nu\}\] contains a club set.

Proof. We prove the lemma by induction on \(\nu\). For \(\nu < \omega_1\) or \(\nu\) a successor ordinal, the induction step is trivial.

Case 1: \(\nu\) is a limit ordinal of cofinality \(\omega\). Let \(\nu_n\), \(n < \omega\), be increasing and cofinal in \(\nu\). By induction hypothesis, \(\{\alpha: A_\alpha \cap \nu_n \text{ is closed in } E \cap \nu_n\}\) contains a club set. Then \(\{\alpha: A_\alpha \text{ is closed in } E \cap \nu\} = \bigcap\{\alpha: A_\alpha \cap \nu_n \text{ is closed in } E \cap \nu_n\}\) contains a club set.

Case 2: \(\nu\) is a limit ordinal of cofinality \(\omega_1\). By c) we can find \(\{\nu_\alpha: \alpha < \omega_1\}\) continuous, increasing, cofinal in \(\nu\), and disjoint from \(E\). If \(A_\alpha \cap \nu\) is not closed in \(E \cap \nu\), define \(h(\alpha)\) to be the least ordinal such that \(A_\alpha \cap \nu_{h(\alpha)}\) is
not closed in \( E \cap v \). Using the regularity of \( \omega_1 \), the hypothesis that each \( A_\alpha \) is countable, and \( f) \), we can find a club set \( C \) of limit ordinals such that if \( \gamma \in C \) and \( \alpha < \gamma \), then \( A_\alpha \subseteq v_\gamma \) and \( h(\alpha) \) is either undefined or less than \( \gamma \). Using \( g) \), for \( \gamma \in C \), \( A_\gamma \subseteq v_\gamma \), hence any limit point of \( A_\gamma \) in \( E \) is less than \( v_\gamma \) (not equal to \( v_\gamma \neq E \)). Hence \( h(\gamma) \) is either undefined or \( h(\gamma) < \gamma \).

If \( h \) presses down on a stationary set, then by the Pressing Down Lemma \( h(\alpha) = \beta \) for some \( \beta \) and stationarily many \( \alpha \)'s. Then the lemma fails for \( v_\beta \) and \( \{ A_\alpha \cap v_\beta : \alpha < \omega_1 \} \), contradicting the inductive hypothesis.

We now define our desired forcing, \( P_{\omega_3} \), by inductively defining notions of forcing \( P_\beta, \beta \leq \omega_3 \). Simultaneously, we will show that \( P_\beta \) is \( \omega_2 \)-cc and \( \omega \)-Baire (i.e. adds no \( \omega \)-sequences of ordinals), so that we may require \( j) \) and \( k) \) below. Explicitly, by induction on \( \beta \leq \omega_3 \), we define \( P_\beta \) to be the set of \( p \) satisfying

\[ h) \quad p \] is a function with domain \( \beta \)

\[ i) \quad p(\alpha) \in P(H_\alpha, K_\alpha) \text{ where } H_\alpha, K_\alpha \text{ are terms for disjoint subsets of } Y \text{ in the forcing language for } P_\alpha \]

\[ j) \quad \{(H_\alpha, K_\alpha) : \alpha < \omega_3 \} \text{ enumerates all terms for disjoint subsets of } Y \text{ in the language for } P_{\omega_3} \]

\[ k) \quad p(\alpha) \in L \text{ (the ground model) (i.e. it is not a term for an element of } P(H_\alpha, K_\alpha) \text{, it is an element of } P(H_\alpha, K_\alpha)) \]

\[ l) \quad p(\alpha) = (\emptyset, \emptyset) \text{ for all but countably many } \alpha \text{'s} \]

\[ m) \quad p \leq q \text{ if } p(\alpha) \supseteq q(\alpha) \text{ for all } \alpha < \beta. \]

That \( P \) has \( \omega_2 \)-cc follows from the continuum hypothesis, \( l) \), and the Delta System Lemma. So \( j) \) is possible.
Aiming towards showing that $P_\beta$ is $\omega$-Baire, let

\[ \{D_n : n \in \omega\} \] be a countable set of dense open subsets of $P_\beta$, and $p$ an arbitrary element of $P_\beta$. Let $N$ be a structure containing everything relevant, e.g. $N = \langle V_{\omega_4}, \in, P_\beta, |P_\beta|, E, \beta, \{D_n : n \in \omega\} \rangle$. Let $N_\rho$, $\rho < \omega_1$, be a continuous increasing sequence of countable elementary submodels of $N$ satisfying $N_\rho \in N_{\rho+1}$. Set $\omega_2 \cap \cup\{N_\rho : \rho < \omega_1\} = \nu$, an ordinal less than $\omega_2$. Applying Lemma 1 to $E$, $\nu$, \{\*\}, we can find $\rho_n$, $n \in \omega$, $\sup\{\rho_n : n \in \omega\} = \rho$, such that $\nu \cap N_\rho$ is closed in $E \cap \nu$.

We define a sequence $\{P_n : n \in \omega\}$ of forcing conditions satisfying

- $P_0 = P$, $P_{n+1} \subset P_n$
- $P_{n+1} \in D_n \cap N_\rho$
- $s(P_{n+1}(a)) \supseteq N_\rho \cap E$, when $P_{n+1}(a) \neq (\phi, \phi)$

Define $q$ with domain $\beta$ by $q(a) = \cup\{P_n(a) ; n \in \omega\}$.

Clearly $q$ satisfies h), k), and l), and $q$ satisfies i) by n) and q), so $q \in P_\beta$. We have found $q$, $q < p$ and $q \in \cap\{D_n : n \in \omega\}$ and may conclude that $P_\beta$, $\beta \leq \omega_3$, is $\omega$-Baire. This completes the simultaneous definition of $P_\beta$ and verification of $\omega_2$-cc and $\omega$-Baire.

In the extension by $P_{\omega_3}$, $X$ is normal. For it is sufficient to consider disjoint $H$ and $K$ subsets of $Y$, and by j) there is a generic pair of open sets separating them. The Continuum Hypothesis is preserved by the extension because it is $\omega$-Baire.

2. Proof of Theorem 2

We imitate Baumgartner [B]. Let $\kappa$ be weakly compact in
M, the ground model, and let $P(\kappa, \omega_2)$ be the Levy forcing collapsing $\kappa$ to $\omega_2$. Let $X^0$ be the name of a locally countable, $<\omega_1$-collectionwise Hausdorff space with \{${}\_a: a < \kappa$\} a closed discrete subset of $X^0$ that can not be screened. We may assume that $X^0 \subseteq V_\kappa$, by $\Pi_1^1$ indescribability, there is a $\lambda < \kappa$ with the same properties. Explicitly, $X^0 \cap V_\lambda$ is the name in the language for $P(\lambda, \omega_2)$ of a locally countable, $<\omega_1$-collectionwise Hausdorff space with \{${}\_a: a < \lambda$\} a closed discrete subset of $X^0 \cap V_\lambda$ that can not be screened.

Let $G$ be an $M$-generic ultrafilter on $P(\lambda, \omega_2)$. We will work in $M^1 = M[G]$, where $\omega_2 = \lambda$, $X$ is $<\omega_1$-collectionwise Hausdorff, and $Y = \{y_\alpha: a < \lambda\}$ witnesses that $X$ is not $<\omega_2$-collectionwise Hausdorff. For each $\alpha < \kappa$ we choose a countable neighborhood $B_\alpha$ of $y_\alpha$, fixed throughout this section. Set $W_\beta = \cup\{B_\alpha: a < \beta\}$.

Lemma 2. There are $S$, $h$ such that

1) $S$ is a stationary subset of $\omega_2$
2) $\delta \in S$ implies that $\text{cf } \delta = \omega$
3) $h: S \rightarrow \omega_2$, $h(\delta) \geq \delta$
4) $y_{h(\delta)} \in \text{closure } W_\delta$

Proof. It suffices to find a set $S$ satisfying 1), 2) and
5) for $\delta \in S$, closure $W_\delta \cap \{y_\alpha: \delta \leq a < \omega_2\} \neq \emptyset$.

Aiming for a contradiction, we assume that there is no such set $S$. Specifically, we assume that there is a club set $C$ such that for $\delta \in C_0 = \{\delta \in C: \text{cf } \delta = \omega\}$, closure $W_\delta \cap \{y_\alpha: \delta \leq a < \omega_2\} = \emptyset$. Let $C'$ be the set of limit points of $C_0$; $C'$ is a club set. We claim

6) for $\delta \in C'$, closure $W_\delta \cap \{y_\alpha: \delta \leq a < \omega_2\} = \emptyset$. 
There are two cases. First, if $\delta \in C'$, cf $\delta = \omega$, 6) holds because $\delta \in C_0$. Second, if $\delta \in C'$, cf $\delta > \omega$ we show 6) using the fact that $X$ is locally countable. If there were $y \in \text{closure } W_\delta \cap \{y_\alpha : \delta \leq \alpha < \omega_2\}$, then $y \in \text{closure } W_\gamma$ for cofinally many $\gamma$ in $\delta$, in particular for some $\gamma \in C_0$, contradiction.

Let $(\gamma(v) : v < \omega_2)$ be the natural, monotone increasing enumeration of $C'$. Define $U_\gamma = W_\gamma(\gamma + 1)$-closure $W_\gamma(v)$; set $U = \{U_\gamma : v < \omega_2\}$. By definition, $U$ is a disjoint family of open sets, each containing at most $\omega_1$ points of $Y$. By 6) $U$ covers $Y$. Using that $X$ is $\leq \omega_1$-collectionwise Hausdorff, we can improve $U$ to screen $Y$. This contradiction establishes Lemma 2.

Note that $P(\kappa, \omega_2) = P(\lambda, \omega_2) \oplus P'$, where $P'$ is countably closed. Our goal is to show that $P'$ does not add a screening of $Y$. Since in the extension $Y = \{y_\alpha : \alpha < \lambda\}$ has cardinality $\omega_1$, we will have shown that $X^0$ is not $\leq \omega_2$-collectionwise Hausdorff, a contradiction. Towards this goal, suppose that $p \in P'$ forces that $\{V_\alpha : \alpha < \lambda\}$ screens $Y$.

Working in $M^1$, let $N = \langle V_{\kappa + \omega}, P', \models p, p, \{y_\alpha : \alpha < \kappa\}, X^0, \{B_\alpha : \alpha < \kappa\}\rangle$. Define a continuous increasing sequence $N_\rho, \rho < \omega_2$, of elementary submodels of $N$ satisfying $\omega_1 \subseteq N_0$, card $N_\rho = \omega_1$, $W_\rho \subseteq N_\rho$. Set $\delta_\rho = N_\rho \cap \lambda$. Then $\{\delta \in \lambda : \delta = \delta_\delta\}$ is a club set in $\lambda$, so there is such a $\delta$ in $S$. Let $B_h(\delta) \cap N_\delta = Z_{n^1} : n \in \omega$.

We define a sequence $p_n, n \in \omega$, of forcing conditions as follows. Set $p_0 = p$; let $p_{n+1} \in N_\delta$ decide $z_n$—either $z_n \not\in \{V_\alpha : \alpha < \lambda\}$ or $z_n \in V_\alpha$ for some specific $\alpha$. The point is that this specific $\alpha$ must be in $N_\delta$, and thus can not be $h(\delta)$. Set $q = \cup\{p_n : n \in \omega\}$; $q$ might not be in $N_\delta$, but $q$ is
in $P'$. Let $q' \supseteq q$ choose $V_h(\delta)$. Because $P'$ is countably closed, $V_h(\delta) \cap B_h(\delta) \in M'$. By our choice of $p_n$'s $V_h(\delta) \cap B_h(\delta) \cap N_\delta = \emptyset$. As $W_\delta \subseteq N_\delta$, $V_h(\delta) \cap B_h(\delta)$ is an open neighborhood of $y_h(\delta)$ demonstrating that $y_h(\delta) \notin$ closure $W_\delta$. We chose $\delta \in S$, so this contradicts 4). This contradiction completes the proof of Theorem 2.

The proofs of the variants of Theorem 2 are parallel and so omitted.

Biblography


[F] W. Fleissner, On $\lambda$-collectionwise Hausdorff spaces, this volume.


[S'] ______, in preparation.

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