BOXES OF COMPACT ORDINALS

by

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If \{X_n : n \in \omega\} is a family of spaces, then \(\square_{n \in \omega} X_n\), called the box product of those spaces, denotes the cartesian product of the sets with the topology generated by all sets of the form \(\prod_{n \in \omega} G_n\), where each \(G_n\) need only be open in the factor space \(X_n\). If \(X_n = X \forall n \in \omega\), we denote \(\square_{n \in \omega} X_n\) by \(\square^\omega X\).

M. E. Rudin [5] and K. Kunen [3 and 6, pg. 58] have shown that CH implies \(\bigcup_{n \in \omega}(\lambda_n + 1)\) is paracompact for every countable collection of ordinals \{\lambda_n : n \in \omega\}. At the 1976 Auburn University Topology Conference I demonstrated [7] that the paracompactness of \(\bigcup^\omega (\omega + 1)\) is implied by the existence of a \(k\)-scale in \(\omega^\omega\), a set-theoretic axiom which is a consequence of, but not equivalent to, Martin's Axiom, and hence CH. In addition, I proved \(\bigcup^\omega(\omega_1 + 1)\) is paracompact iff \(\bigcup^\omega(\alpha + 1)\) is paracompact \(\forall\) countable ordinals \(\alpha\). If this is coupled with E. van Douwen's (\(\exists\) a \(k\)-scale in \(\omega^\omega\) \Rightarrow \(\square_{n \in \omega} X_n\) is paracompact for all collections \(\{X_n : n \in \omega\}\) of compact metrizable spaces [1], we have \(\bigcup^\omega(\omega_1 + 1)\) is paracompact if \(\exists\) a \(k\)-scale in \(\omega^\omega\). However, none of the proofs generalize to higher ordinals (\(\bigcup^\omega(\omega_2 + 1)\), for example). We conjecture:

\textit{If} \(\bigcup^\omega(\omega + 1)\) \textit{is paracompact, then} \(\bigcup^\omega(\lambda + 1)\) \textit{is paracompact} \(\forall\) ordinals \(\lambda\).\(^1\)

\(^1\text{It is unknown whether it is consistent for} \bigcup^\omega(\omega + 1)\text{ not to be paracompact; however,} \exists\text{ compact spaces} X_n\text{ such that} \bigcup_{n \in \omega} X_n\text{ is not normal. Moreover, irrationals} x(\bigcup^\omega(\omega + 1))\text{ is not normal [6, pg. 58].}
Toward this conjecture we show:

Suppose $\lambda$ is an ordinal for which $\bigcup_{n \in \omega} (\lambda_n + 1)$ is paracompact whenever $\lambda_n < \lambda \forall n \in \omega$, then $\bigcup_\omega (\lambda + 1)$ if either of the following holds:

1. $\text{cf}(\lambda) \neq \omega$ (Theorem 1).
2. $\text{cf}(\lambda) = \omega$ and there is a $\kappa$-scale in $\omega_\omega$ (Theorem 2).

Now suppose $\{X_n : n \in \omega\}$ is a family of sets and for each $f \in \prod_{n \in \omega} X_n$,

$$E(f) = \{g \in \prod_{n \in \omega} X_n : (\exists m \in \omega) n > m \Rightarrow g(n) = f(n)\},$$

then $\{E(f) : f \in \prod_{n \in \omega} X_n\}$ forms a partition of $\prod_{n \in \omega} X_n$ and the resultant quotient set is denoted by $\nabla_{n \in \omega} X_n$. If $S \subseteq \prod_{n \in \omega} X_n$, we let $E(S)$ denote its image in $\nabla_{n \in \omega} X_n$.

**Lemma** (Kunen [3 and 6, pg. 58]). Suppose $X_n$ is a compact Hausdorff space for each $n \in \omega$ and $\nabla_{n \in \omega} X_n$ has the quotient topology induced by $\bigcup_{n \in \omega} X_n$, then

1. $G_\delta$-sets in $\nabla_{n \in \omega} X_n$ are open
2. $\bigcup_{n \in \omega} X_n$ is paracompact iff $\nabla_{n \in \omega} X_n$ is paracompact
3. Every open cover of $\nabla_{n \in \omega} X_n$ has a subcover of cardinality $\leq c$ (the cardinality of the continuum) whenever $X_n$ is scattered $\forall n \in \omega$.

For $A, B \in \mathcal{P}(\omega)$ define $A \leq B$ if $A - B$ is finite; $A \equiv B$ if $A \leq B$ and $B \leq A$. Observe that $\equiv$ is an equivalence relation on $\mathcal{P}(\omega)$. Suppose $\lambda$ is an ordinal and $f \in \omega^\lambda$, for each $A \in \mathcal{P}(\omega)$, we define in $\nabla^\omega(\lambda + 1)$, $\langle A, f \rangle = E(\prod_{n \in \omega} A f(n))$, where $E(\cdot)$ denotes an image.
\[ A_f(n) = \begin{cases} \{f(n) + 1, \lambda\} & \text{if } n \in A \\ \{0, f(n)\} & \text{if } n \notin A. \end{cases} \]

\[ \{\langle A, f\rangle : A \in \mathcal{P}(\omega)\} \text{ forms a clopen partition of } \nu^\omega(\lambda + 1) \]
since \( A \equiv B \text{ iff } \langle A, f\rangle \cap \langle B, f\rangle \neq \emptyset. \]

**Theorem 1.** Suppose \( \lambda \) is an ordinal with \( \text{cf}(\lambda) \neq \omega \), then for \( \nu^\omega(\lambda + 1) \) to be paracompact it is necessary and sufficient that \( \nu^\omega(\alpha + 1) \) be paracompact \( \forall \alpha < \lambda. \)

**Proof.** Necessity is obvious so we prove sufficiency only.

Without loss of generality, we assume \( \lambda \) is the supremum of an increasing sequence \( \{n_\alpha : \alpha < \text{cf}(\lambda)\} \). Let \( R \) be an open cover of \( \nu^\omega(\lambda + 1) \). For each \( \tau < \omega \) and \( d \in \tau^\omega \) we construct inductively \( V(d), W(d), \theta(d), \) and \( A(d) \) to satisfy:

1. \( V(d) \) and \( W(d) \) are clopen subsets of \( \nu^\omega(\lambda + 1) \), \( \exists U \in R \ni V(d) \subseteq U, V(d) \cup W(d) \subseteq W(d \uparrow \sigma) \forall \sigma < \tau, \) and if \( \sigma < \tau \) is a limit ordinal, then \( W(d \uparrow \sigma) = \cap_{\rho<\sigma} W(d \uparrow \rho). \)

2. If \( \sigma \leq \tau \) is an odd ordinal \( \sigma \), then \( \{V(e) : \text{dom}(e) \leq \sigma\} \cup \{W(e) : \text{dom}(e) = \sigma\} \) is a pairwise-disjoint covering of \( \nu^\omega(\lambda + 1). \)

3. \( A(d) \) is an infinite subset of \( \omega \) and if \( \sigma \leq \tau \) is a non-limit ordinal, then \( A(d \uparrow \sigma) \leq A(d \uparrow \rho) \forall \rho < \sigma. \)

4. If \( E(x) \in W(d) \) and \( \phi < A \leq A(d \uparrow \sigma) \forall \sigma < \tau, \) then \( E(\{y : x(n) \leq y(n) \leq \lambda \text{ if } n \in A, y(n) = x(n) \text{ if } n \notin A\}) \subseteq W(d). \)

5. \( \theta(d) \in \omega^\lambda \) is a constant function with values in \( \{n_\alpha : \alpha < \text{cf}(\lambda)\} \) and if \( \sigma \leq \tau \) is even, then \( \theta(d \uparrow \sigma)(0) > \theta(d \uparrow \rho)(0) \forall \rho < \sigma. \)

\( \sigma \) is an odd ordinal when \( \sigma = \sigma_0 + 2n + 1, \) where \( \sigma_0 = 0 \) or is a limit ordinal and \( n \in \omega. \) If \( \sigma \) is not odd it is even.
(6) If $\sigma \leq \tau$ is odd, then $W(d \uparrow \sigma) \leq \langle A(d \uparrow \sigma), \theta(d \uparrow \sigma) \rangle$.

(7) If $\sigma \leq \tau$ is a non-limit even ordinal and $\rho = \sigma - 1$, then there exists a clopen subset $G(d \uparrow \sigma)$ of $\bigvee_{n \in A(d \uparrow \rho)} (\theta(d \uparrow \sigma)(n) + 1)$ such that

\[ V(d \uparrow \sigma) = W(d \uparrow \sigma) \cup \langle A(d \uparrow \rho), \theta(d \uparrow \sigma) \rangle \text{ and} \]

\[ W(d \uparrow \sigma) = \{ E(x) \in W(d \uparrow \rho) : E(x \uparrow \omega - A(d \uparrow \rho)) \in G(d \uparrow \sigma) \}. \]

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (7) for all $d \in T \subset \tau$. Suppose for an ordinal $\rho < \omega_1$ we have constructed $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy (1) through (7) for all $d \in T \subset \tau$. Our construction at $\rho$ needs three cases:

Case 1. $\rho$ is an odd ordinal

Let $\tau = \rho - 1$ and $\theta(e) = \theta(d)$ if $e \in \mathcal{P} c$ and $e \uparrow \tau = d$. Let
be a listing of exactly one element chosen from each equivalence class of elements of

\[ \{A: \phi < A < A(d \uparrow 0), \sigma < \tau\}. \]

For each \( e \in \rho \) we let

\[ W(e) = W(e \uparrow \tau) \cap \langle A(e), \theta(e) \rangle. \]

If \( d \in \tau \) and \( W(d) \cap \langle \phi, \theta(d) \rangle \) is a clopen subset of \( E\{0, \theta(d)(0)\} \); therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of \( R \)

\[ \{V(e): e \in \rho, e \uparrow \tau = d\} \] whose union is \( W(d) \cap \langle \phi, \theta(d) \rangle \).

Clearly (1) through (7) are satisfied.

Case 2. \( \rho \) is a non-limit even ordinal.

Let \( \sigma = \rho - 1 \) and \( A(e) = A(d) \) if \( e \in \rho \) and \( e \uparrow \tau = d \). If \( d \in \tau \) and \( W(d) = \emptyset \), we let \( W(e) = V(e) = \emptyset \) and

\[ \theta(e)(n) = n_\alpha \text{ if } \theta(d)(n) = n_{\alpha-1} \quad \forall n \in \omega \]

If \( d \in \tau \) and \( W(d) \neq \emptyset \), let

\[ Y^*(d) = \{g: g^{-1}(\lambda) = A(d), E(g) \in W(d)\}. \]

We will wish to cover \( Y^*(d) \) by

\[ \cup \{W(e): e \uparrow \tau = d\}. \]

From (4), \( Y(d) = \{g \uparrow \omega - A(d): g \in Y^*(d)\} \neq \emptyset \).

In \( \bigvee_{\not\in \not A(d)} (\theta(d)(n) + 1) \), let

\[ R(d) = \{E(\Pi_{n \in \omega A(d)} U(n)): E(\Pi_{n \in \omega} U(n)) \subseteq \text{some } U \in R, \]

\[ E(\Pi U(n)) \cap Y^*(d) \neq \emptyset \}. \]

From (5) of the induction hypothesis and the lemma, (ii) and (iii), \( \exists \) a pairwise disjoint clopen refinement \( \{G(\gamma): \gamma < \omega\} \) of \( R(d) \) whose union is \( E(Y(d)) \). If \( e \in \rho \), \( e \uparrow \tau = d, e(\tau) = \gamma \), then let

\[ W(e) = \{E(x) \in W(d): E(x \uparrow \omega - A(d)) \in G(\gamma)\}. \]
For each \( \gamma \) we may find \( n_a(\gamma) > \theta(d)(0) \) such that
\[
\{E(x) \in W(d) : E(x \uparrow \omega - A(d)) \in G(\gamma) \text{ and } x(n) > n_a(\gamma) \} \forall \text{ but finitely many } n \in A(d) \subseteq \text{ some } U \in R.
\]

Let \( \theta(e)(n) = n_a(\gamma) \forall n \in w \) and \( V(e) = W(e) \cap \langle A(d), \theta(e) \rangle \).

Certainly (1) through (7) are satisfied.

\textbf{Case 3.} \( \rho \) is a limit ordinal.

If \( e \in \rho \), let \( A(e) = \omega, V(e) = \emptyset \), and find the first \( \alpha < \omega_1 \ni n_a > \theta(e \uparrow \tau)(0) \forall \tau < \rho \). We choose \( \theta(e)(n) = n_a \forall n \in w \). To satisfy (1) through (7) we observe that (i) of the lemma allows
\[
W(e) = \bigcap_{\tau < \rho} W(e \uparrow \tau)
\]
to be clopen.

The proof to Theorem 1 is completed.

If \( \omega \omega \) is ordered by \( f < g \) if \( \{n : g(n) \leq f(n)\} \not= \emptyset \), then for an ordinal \( k \), a \( k \)-scale is an order-preserving injection \( s : k + \omega \omega \text{ such that } \{s(\alpha) : \alpha < k\} \text{ is cofinal in } \omega_\omega \). Recall [2,7] that CH \( \Rightarrow \exists \) an \( \omega_1 \)-scale; MA \( \Rightarrow \exists \) a c-scale; an \( \omega \)-scale; \( \exists \) a \( k \)-scale and \( \ell \)-scale \( \Rightarrow \) cf(\( k \)) = cf(\( \ell \)); for every model \( m \) with regular ordinals \( k \) and \( \ell \) with cf(\( k \)) \( \not= \omega \not= \text{ cf}(\ell) \) and \( k \leq \ell \), there is a model \( n \supseteq m \) with a \( k \)-scale in \( \omega \omega \) and \( c = \ell_l \) and \( \exists \) models \( m \) of ZFC without \( k \)-scales for any \( k \).

\textbf{Theorem 2.} ( \( \exists \) a \( k \)-scale in \( \omega \omega \)). Suppose cf(\( \lambda \)) = \( \omega \), then for \( L^{\omega}(\lambda + 1) \) to be paracompact it is necessary and sufficient that \( \exists \{\gamma_n : n \in \omega\} \subseteq \lambda \ni \sup n \in \omega \gamma_n = \lambda \) and \( L^{\omega}(\gamma_n + 1) \) is paracompact.

\textbf{Proof.} Necessity is obvious so we prove sufficiency.

WLOG assume \( \gamma_n < \gamma_{n+1} \forall n \in \omega \), cf(\( \gamma_n \)) = \( l \) \( \forall n \in \omega \), and \( \{s(\alpha) : \alpha < k\} \) is a \( k \)-scale in \( \omega \omega \) for a regular \( k \). Let \( R \) be
an open cover of $\mathcal{V}^\omega(\lambda + 1)$. For each $\tau < \kappa$ and $d \in T_c$ we construct inductively $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ to satisfy:

1. $V(d)$ and $W(d)$ are clopen subsets of $\mathcal{V}^\omega(\lambda + 1)$, $\exists U \in R \exists V(d) \subseteq U$, $V(d) \cup W(d) \subseteq W(d \uparrow \sigma)$ $\forall \sigma < \tau$, and if $\sigma < \tau$ is a limit ordinal $W(d \uparrow \sigma) = \cap_{\rho < \sigma} W(d \uparrow \rho)$.

2. If $\sigma \leq \tau$ is an odd ordinal, then
   $$\{V(e): \text{dom}(e) \leq \sigma\} \cup \{W(e): \text{dom}(e) = \sigma\}$$
   is a pairwise-disjoint covering of $\mathcal{V}^\omega(\lambda + 1)$.

3. $A(d)$ is an infinite subset of $\omega$ and if $\sigma \leq \tau$ is a non-limit ordinal, then $A(d \uparrow \sigma) \subseteq A(d \uparrow \rho)$ $\forall \rho < \sigma$.

4. $\theta(d)(n) = \gamma_{S(a)}(n)$ $\forall n \in \omega$ and some $a < \kappa$; and if $\sigma \leq \tau$ is even, then
   $$\{n: \theta(d \uparrow \sigma)(n) \leq \theta(d \uparrow \rho)(n)\} \equiv \phi$$
   $\forall \rho < \sigma$.

5. If $\sigma \leq \tau$ is odd, then $W(d \uparrow \sigma) \subseteq \langle A(d \uparrow \sigma), \theta(d \uparrow \sigma) \rangle$ and
   $$\{V(e): e \in T_c, e \uparrow \sigma - 1 = d \uparrow \sigma - 1\} = \langle \phi, \theta(d \uparrow \sigma) \rangle \cap W(d \uparrow \sigma - 1).$$

6. If $\sigma \leq \tau$ is a non-limit even ordinal, then
   $$V(d \uparrow \sigma) = W(d \uparrow \sigma) \cup \langle A(d \uparrow \sigma - 1), \theta(d \uparrow \sigma - 1) \rangle.$$

Now suppose our objects $V(d)$, $W(d)$, $\theta(d)$, and $A(d)$ have been constructed to satisfy (1) through (6) $\forall d \in T_c$ $\forall \tau < \kappa$.

For $x \in \Pi^\omega(\lambda + 1)$ define
$$x^\#(n) = \begin{cases} 0 & \text{if } x(n) = \lambda \\ x(n) & \text{otherwise.} \end{cases}$$

We may find the first $a \ni \{n: \gamma_{S(a)}(n) \leq x^\#(n)\} = \phi$. If $a = a_0 + m$, where $a_0 = 0$ or is a limit ordinal and $m \in \omega$, let $\tau = a_0 + 2(m + 1)$. From (2), (4), (5), and (6) we have
$$E(x) \in \cup\{V(e): \text{dom}(e) \leq \tau\}.$$

Therefore, $\{V(d): d \in T_c, \tau < \kappa\}$ is a pairwise-disjoint clopen refinement of $R$ covering $\mathcal{V}^\omega(\lambda + 1)$. So we must complete our
construction.

Let \( A(\phi) = \omega \), \( W(\phi) = \gamma^\omega(\lambda + 1) \), and \( \alpha \) be the first ordinal such that \( E(\prod \mathbb{N}^{\omega} [\gamma_{S(\alpha)}(n), \lambda]) \) is contained in some \( U \in \mathcal{R} \). Let \( \theta(\phi)(n) = \gamma_{S(\alpha)}(n) \) \( \forall n \in \omega \) and \( V(\phi) = \langle A(\phi), \theta(\phi) \rangle \).

Suppose for an ordinal \( \rho < \kappa \) we have constructed \( V(d) \), \( W(d) \), \( \theta(d) \), and \( \Lambda(d) \) to satisfy (1) through (6) \( \forall d \in \mathcal{T} \) \( \forall \tau < \rho \). Our construction at \( \rho \) needs three cases:

Case 1. \( \rho \) is an odd ordinal.

Let \( \tau = \rho - 1 \) and \( \theta(e) = \theta(d) \) if \( e \in \mathcal{P} \) and \( e \uparrow \tau = d \). Let

\[ \{ A(e) : e \in \mathcal{P}, e \uparrow \tau = d \} \]

be a listing of exactly one element from each equivalence class of elements of

\[ \{ A : \phi < A < A(d \uparrow \sigma), \sigma \leq \tau \} \].

For each \( e \in \mathcal{P} \) we let

\[ W(e) = W(e \uparrow \tau) \cap \langle A(e), \theta(e) \rangle \].

If \( d \in \mathcal{T} \), then \( W(d) \cap \langle \phi, \theta(d) \rangle \) is a clopen subset of

\[ E(\prod \mathbb{N}^{\omega} [0, \theta(d)(n)]) \]

and \( \prod \mathbb{N}^{\omega} [0, \theta(d)(n)] \) is a clopen subset of a subproduct of \( \prod \mathbb{N}^{\omega} (\gamma_n + 1) \); therefore, by the lemma (ii) and (iii) we may find a pairwise-disjoint clopen refinement of \( R, \{ V(e) : e \in \mathcal{P}, e \uparrow \tau = d \} \) whose union is \( W(d) \cap \langle \phi, \theta(d) \rangle \). Clearly, (1) through (6) are satisfied.

Case 2. \( \rho \) is a non-limit even ordinal.

Let \( \tau = \rho - 1 \), and \( A(e) = A(d) \), and \( W(e) = W(d) \) if \( e \in \mathcal{P} \) and \( e \uparrow \tau = d \). If \( d \in \mathcal{T} \) and \( W(d) = \phi \), we let \( W(e) = V(e) = \phi \) and

\[ \theta(e)(n) = \gamma_{S(\alpha + 1)}(n) \] \( \forall n \in \omega \); where

\[ \theta(e \uparrow \tau)(n) = \gamma_{S(\alpha)}(n) \] \( \forall n \in \omega \).
If $\varepsilon \in \mathcal{T}$, $W(d) \neq \emptyset$, and

$$Y(d) = \{g \uparrow \omega - A(d): \gamma^{-1}(\lambda) = A(d), E(y) \in W(d)\} = \emptyset.$$ 

In $\bigwedge_{n \in \mathcal{A}(d)} (\theta(d)(n) + 1)$, let

$$R(d) = \{E(\Pi_{n \notin \mathcal{A}(d)} U(n)): E(\Pi_{n \in \omega} U(n)) \subseteq \text{some } U \in R,$$

$$\exists E(g) \in W(d) \cap E(\Pi_{n \in \omega} U(n)), g^{-1}(\lambda) = A(d)\}.$$ 

Since $\bigwedge_{n \notin \mathcal{A}(d)} (\theta(d)(n) + 1)$ is homeomorphic to a clopen subset of a subproduct of $\bigwedge_{n \in \omega} (\gamma_n + 1)$, we may use the lemma, (ii) and (iii), to find a pairwise disjoint clopen refinement $\{G(\delta): \delta < \omega\}$ of $R(d)$ whose union is $E(Y(d))$. If $e \in \mathcal{O} \cap c$, $e \uparrow \tau = d$, $e(\tau) = \delta$, then let $\alpha(\delta)$ be the first ordinal $> \alpha(d)$, where $\theta(d)(n) = \gamma_s(\alpha(d))(n) \forall n \in \omega$, such that

$$V(e) = \{E(x) \in W(d): E(x \uparrow \omega - A(d)) \subseteq G(\delta), x(n) >$$

$$\gamma_s(\alpha(\delta))(n) \forall n \in \omega\}$$

is contained in a member of $R$. Let $\theta(e)(n) = \gamma_s(\alpha(\delta))(n) \forall n \in \omega$. Clearly, (1) through (6) are satisfied.

Case 3. $\rho$ is a limit ordinal.

If $e \in \mathcal{O} \cap c$, let $A(e) = \omega$, $V(e) = \emptyset$, and $\theta(e)(n) = \gamma_s(\alpha(n)) \forall n \in \omega$, where

$$\alpha = \sup\{\beta: \theta(e \uparrow \tau)(n) = \gamma_s(\beta)(n) \forall n \in \omega, \tau < \rho\}.$$ 

To see that (1) through (6) are satisfied, we must show

$$W(e) = \bigcap_{\tau < \rho} W(e \uparrow \tau)$$

is open. However, if $E(x) \in W(e)$, then the induction hypothesis and the definition of $W(d)$ in Case 2 yields

$$E(\Pi_{n \in \omega} [x^*(n), x(n)]) \subseteq W(e),$$

where

$$x(n) \text{ if } \text{cf}(x(n)) = 1$$

$$x^*(n) = \begin{cases} x(n) + 1 \text{ if } x(n) \text{ is a limit } > \theta(e)(n) \\ \sup\{\theta(e \uparrow \tau)(n): \theta(e \uparrow \tau)(n) < x(n), \tau < \rho\} + 1, \text{ otherwise.} \end{cases}$$
This completes the construction and the proof of Theorem 2.

Remarks

A. There are many models of ZFC, constructed via forcing, in which there are no \( k \)-scales \([2]\). However, J. Roitman \([4]\) has shown that in some of these models, techniques inadvertently, in some sense, yield \( \prod_{n \in \omega} X_n \) paracompact \( \forall \) compact metrizable \( X_n \); specifically she has shown:

In a model \( m \) of set theory which is a direct iterated CCC extension of length \( k \) of a model \( n \), \( \text{cf}(k) > \omega \) \( \implies \nabla_{n \in \omega} X_n \) is paracompact if \( X_n \) is regular and separable.

A simple adaptation of her proofs will give the conclusion of Theorem 2 in \( m \).

B. Suppose \( u_0 \) is an ordinal and for \( n > 0 \) \( u_n \) is the lexicographic ordered product of \( u_{n-1} \) with itself. Let \( u = \sup_{n \in \omega} u_n \). It is unknown whether \( \exists \) a \( k \)-scale in \( \omega_{\omega} \) \( \implies \bigsqcup^\omega (u + 1) \) is paracompact when \( u_0 = \omega_1 \); however, our theorems show \( \exists \) a \( k \)-scale in \( \omega_{\omega} \) \( \implies \bigsqcup^\omega (u_n + 1) \) is paracompact \( \forall n \in \omega \). It is unknown whether \( \bigsqcup^\omega (u + 1) \) is paracompact \( \implies \bigsqcup^\omega (u_n + 1) \) is paracompact when \( u_0 = \omega \); although \( \bigsqcup^\omega (u_n + 1) \) is paracompact for each \( n \). The simplest question still unanswered is "Does there exist a model \( m \) of ZFC in which \( \bigsqcup^\omega (\lambda + 1) \) is not paracompact for some ordinal \( \lambda ? \)" The hardest question asks that \( \lambda = \omega \).

C. We observe a recent result communicated to the author by E. K. van Douwen: If \( X_n \) is compact \( \forall n \in \omega \), then \( \bigsqcup_{n \in \omega} X_n \) is pseudo-normal. The author gives much appreciation to the referee whose suggestions for clarification of

\[ \forall^\omega (u_n + 1) \text{ may be embedded in } \forall^\omega (u + 1) \].
unnecessary technicalities in our proofs appear.

Added in proof

Recently, J. Roitman has proved that \( \bigcap_{n \in \omega} X_n \) is paracompact whenever each \( X_n \) is compact first countable and \( \omega \) fails to have a cofinal family of cardinality less than the continuum. A corollary to this theorem and our theorems 1 and 2 yields \( c = \omega_2 \Rightarrow \square^{(\omega_1 + 1)} \) is paracompact. Independently, I have shown the same corollary and, in addition:

Suppose, in theorem 2, \( \exists \ k\text{-scale in } \omega \) is replaced by \( k \) is the least cardinal of any cofinal family in \( \omega \) and \( A \subseteq \mathcal{P}(\omega) \) with \( |A| = k \), then

\[ E(\{x \in \square^\omega \lambda + 1: x^{-1}(\lambda) \in A\}) \text{ is paracompact.} \]

References

1. E. K. van Douwan, \( \exists \ k\text{-scale implies } \bigcap_{n \in \omega} X_n \) is paracompact if \( X_n \) is compact metric \( \forall n \in \omega \), lecture presented at the Ohio University Conference on Topology, May 1976.


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