IRREDUCIBLY ESSENTIAL MAPS FROM
INVERSE LIMITS

by

GARY A. FEUERBACHER
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By a "continuum" is meant a compact, connected metric space. A "polyhedron" is the space of a finite simplicial complex. A "graph" is a connected, one-dimensional polyhedron. A map from a continuum X into a graph G is "essential" if it is not homotopic to a constant map; it is "irreducibly essential" if it is essential, but its restriction to any closed, proper subset of X is inessential.

If T is a continuum, then there is an inverse system \((P_i,F^{i+1}_i)_{i \in \mathbb{N}}\) with each \(P_i\) a polyhedron and each \(F^{i+1}_i\) a simplicial map, such that \(T = \lim_{\rightarrow}(P_i,F^{i+1}_i)\).

In what follows, suppose \(M\) is a one-dimensional continuum. Let \(M\) be represented as the inverse limit of the inverse system \((X_i,f^{i+1}_i)\), with each \(X_i\) a graph and each bonding map \(f^{i+1}_i\) a simplicial map from \(X_j\) onto \(X_i\). The \(i\)th projection map will be denoted \(\pi_i\).

Notation. In what follows, if \(Y\) is a metric space, \(d_Y\) will denote its metric. For the factor spaces of the inverse system \((X_i,f^{i+1}_i)\), \(d_j\) will be used in place of \(d_{X_j}\). Whenever each of \(f\) and \(g\) is a map from a compactum \(A\) into a compactum \(B\), \(f \equiv g\) will mean "\(f\) is homotopic to \(g\)"; in case \(t > 0\), 
\(|f-g| < t\) means that the distance from \(f\) to \(g\) in the space \(B^A\) is less than \(t\), i.e.,
\[
\text{lub}(d_B(f(a),g(a))) < t.
\]
If \( Y \) is a metric space, and \( P \in Y \), and \( e > 0 \), then \( B(P, e) \) will denote the open ball with center \( P \) and radius \( e \).

The first section of this paper describes certain properties of the inverse system \((X_i, f_{i+1}^i)\) which are related to the existence of an essential map from \( M \) onto the unit circle \( S^1 \).

**Lemma 1.** If \( G \) is a graph and \( k \) is a map from \( M \) into \( G \), then there are a positive integer \( n \) and a map \( h \) from \( X_n \) into \( G \) such that for each \( i \geq 0 \), \( h \circ f_{n+1}^n \circ \pi_{n+i} \) is homotopic to \( k \).

**Proof.** Suppose \( G \) is a graph and \( k \) a map from \( M \) into \( G \).

By Theorem 0 of [1, §54, VIII, p. 379], the components of \( G^M \) correspond to its homotopy classes, and these components are closed-open. Let \( \delta > 0 \) be such that the open ball with center \( k \) and radius \( \delta \) is contained in the component of \( G^M \) that contains \( k \). We may regard \( G \) as \( \text{Lim}(Y_i, t_{i+1}^i) \) with \( Y_i = G \) and \( t_{i+1}^i = \text{Id} \) for each \( i \). By Lemma 1 of [2, p. 39], let \( m \) be a positive integer and \( W \) a map from \( X_m \) into \( G \) such that the diagram

\[
\begin{array}{ccc}
X_m & \xrightarrow{\pi_m} & M \\
\downarrow W & & \downarrow k \\
G & \xrightarrow{\text{Id}} & \text{Id}
\end{array}
\]

is \( \delta \)-commutative. Then \( |k - W \circ \pi_m| < \delta \). Also, since \( \pi_m = f_m^j \circ \pi_j \) for \( j \geq m \), we have

\[ |k - W \circ f_m^j \circ \pi_j| < \delta \]

and thus \( k \approx W \circ f_m^j \circ \pi_j \) for each \( j \geq m \).

We have \( W \) and \( m \) as the needed \( h \) and \( n \), respectively.
Theorem 1. If G is a graph, k: M → G a map, n a positive integer, and h: Xn → G a map such that

\[ h \circ f_{n+1}^n \circ \pi_{n+1} \cong k \]

for each i ≥ 0, then k is essential if and only if for each i ≥ 0, h \( \circ f_{n+1}^n \) is essential.

Proof. (This argument is a modification of the proof for Q9 in [3, p. 82].)

Let G, k, h, and n be as in the Lemma 1. Suppose k is essential. If i ∈ N and h \( \circ f_{n+1}^n \) is inessential, then

\[ (h \circ f_{n+1}^n) \circ \pi_{n+1} \text{ is inessential, a contradiction.} \]

Now suppose h \( \circ f_{n+1}^n \) is essential for each i ∈ N. Suppose k is inessential. Let t = h \( \circ \pi_n \); by the Lemma 1, since t \cong k, t is inessential. Let \( \tilde{G} \) be the universal covering space of G with projection p. Since t is inessential, it may be lifted through \( \tilde{G} \); let t* be a lift of t, and let H = t*(M). Let \( \xi \) be an open cover of H by sets open in \( \tilde{G} \) such that if E ∈ \( \xi \), then p|E is a homeomorphism from E onto p(E) in G. The Lemma Q3 of [3, p. 80] may be modified to read: For any open cover U of M there exists a positive integer j > n and a finite cover \( v \) of \( X_j \) such that \{\pi_j^{-1}(V): V ∈ v\} refines U. The same argument as given by Case and Chamberlin is valid, substituting "X_i" for "B" (representing the figure "8," the union of two circles with a common point).

Let U be the collection of all inverse images under t* of elements of \( \xi \); U = \{t*^{-1}(E): E ∈ \( \xi \)\}. By Q3, let j > n and \( v \) a finite cover of \( X_j \) such that \{\pi_j^{-1}(V): V ∈ v\} refines U.

Let c be the relation, c ⊂ \( X_j × \tilde{G} \), to which the ordered pair (a,b) belongs if and only if there is a point Z ∈ \( \pi_j^{-1}(a) \)
such that \( b = t^*(Z) \).

Now, \( c \) is a function. For: let \((a,b)\) and \((a,b')\) be in \( c \). Let \( b = t^*(Z) \) and \( b' = t^*(Z') \), with \( Z, Z' \in \pi_j^{-1}(a) \). Then
t(\( Z \)) = h(\( \pi_n(Z) \)) = h(f_n^j(\( \pi_j(Z) \))) = h(f_n^j(\( \pi_j(Z') \))) = t(\( Z' \)), hence
\( p \ t^*(Z) = p \ t^*(Z') \). Let \( a \in \Omega \in \nu \); then since \( \{ \pi_j^{-1}(V) : V \in \nu \} \) refines \( U \), there is \( E \in \xi \) such that \( t^*(\pi_j^{-1}(a)) \subseteq E \). But \( p \) is one-to-one on \( E \); hence since \( t^*(Z), t^*(Z') \in E \), and \( p \ t^*(Z) = p \ t^*(Z') \), \( b' = t^*(Z') = t^*(Z) = b \), and \( c \) is single valued.

To show continuity, we note that for \( x \in V \in \nu \), and
\( \pi_j(Z) = x \), \( p \ c(x) = p \ t^*(Z) = t(Z) = h(f_n^j(\pi_j(Z))) = h(f_n^j(x)) \), and \( c(x) = (p|E)^{-1}(h(f_n^j(x))) \) with \( E \) an element of \( \xi \) such that \( t^*(\pi_j^{-1}(V)) \subseteq E \). Since \( h \circ f_n^j \) and \( (p|E)^{-1} \) are continuous on \( V \), so is \( c \). Since \( c \) is continuous on each member of \( \nu \), \( c \) is continuous on \( X_j \).

Also for any \( x \) in \( X_j \), \( p \ c(x) = h \circ f_n^j(x) \), i.e., \( c \) is a lift of \( h \circ f_n^j \) through \( \tilde{G} \). Since \( G \) is a graph, and \( \tilde{G} \) is simply connected, \( c(X_j) \) is contractible, and \( c \) is inessential. Therefore, \( h \circ f_n^j \) is inessential, a contradiction.

**Proposition.** Consider \((S^1,d)\) as a metric space. In what follows, let \( \theta > 0 \) be such that any two points \( a \) and \( b \) of \( S^1 \), with \( d(a,b) < \theta \), are non-antipodal. If \( H \) is a compactum, and each of \( f \) and \( g \) is a map from \( H \) into \( S^1 \), with \( |f - g| < \theta \), then \( f \equiv g \) \([4, \text{p. 85}]\).

**Definition 1.** Suppose \( j \) is a non-negative integer.

Suppose \( C \) is an infinite sequence of simple closed curves. If, for each \( i \geq 1 \),

1. \( C_i \subseteq X_{j+i} \), and
2. \( C_i \subseteq f_n^{j+i+1}(C_{i+1}) \),
and if (3) there exists a map \( h: X_{j+1} \to S^1 \) such that, for each positive integer \( p \), \( h \circ f_{j+1}^{j+p} | C_p \) is essential, then \( C \) will be called an \( M \)-cycle. We will say that \( C \) is associated with the map \( h \).

If, in addition,

(4) there is a map \( k: M \to S^1 \) such that \( |h \circ \pi_{j+1} - k| < \theta/2 \), then \( C \) will be called an \( M \)-cycle on which \( k \) is essential. A finite (or infinite) sequence of simple closed curves having properties (1), (2), (3), and (4) will be said to have property \( p_4 \).

The next result relates the concept of an \( M \)-cycle to the notion of a \( K \)-cycle, with \( K \) being a proper subcontinuum of \( M \); in this case, \( K \) is also an inverse limit, i.e.,

\[
K = \operatorname{Lim}(Y_i, g_i^{i+1}) = \operatorname{Lim}(\pi_i(K), f_i^{i+1} | \pi_{i+1}(K)).
\]

**Lemma 2.** If \( H \) is a subcontinuum of \( M \), \( j \) is a non-negative integer, and \( C \) is a sequence of simple closed curves satisfying, for each \( i \), (1) \( C_i \subset \pi_{j+i}(H) \), (2) \( C_i \subset f_{j+1}^{j+i+1}(C_{i+1}) \), and \( k \) is a map from \( M \) into \( S^1 \), and \( h \) is a map from \( X_{j+1} \) into \( S^1 \) such that \( |h \circ \pi_{j+1} - k| < \theta/2 \), then \( C \) is an \( M \)-cycle on which \( k \) is essential if and only if \( C \) is an \( H \)-cycle on which \( k | H \) is essential.

**Proof.** Suppose \( H, j, C, k, \) and \( h \) are as in the hypothesis. Suppose \( C \) is an \( M \)-cycle on which \( k \) is essential. By definition 1, let \( g \) be a map, \( g: X_{j+1} \to S^1 \), such that

\[
|g \circ \pi_{j+1} - k| < \theta/2,
\]

and, for each \( i \), \( g \circ f_{j+1}^{j+i} \) is essential on \( C_i \). \( |g \circ \pi_{j+1} | H - k | H| < \theta/2 \). Also, \( H = \operatorname{Lim}(\pi_i(H), f_i^{i+1} | \pi_{i+1}(H)) \). We have \( g \circ \pi_{j+i} | H = (g | \pi_{j+1}(H)) \circ \pi_{j+1} | H, \)
and C is an H-cycle on which k | H is essential.

Now suppose C is an H-cycle on which k | H is essential. By definition, let t be a map, t: \( \pi_{j+1}(H) \to S^1 \), such that

\[
|t \circ \pi_{j+1}|H - k|H| < \theta/2,
\]

and, for each i, \( t \circ f_{j+1}^i|C_i \) is essential. We have

\[
|h \circ \pi_{j+1}|H - k|H| < \theta/2,
\]

whence

\[
|t \circ \pi_{j+1}|H - h \circ \pi_{j+1}|H| < \theta.
\]

This implies that

\[
|t - h|\pi_{j+1}(H)| < \theta, \quad \text{and, for each } i, \text{ since}
\]

\[
f_{j+1}^i(C_i) \subset \pi_{j+1}(H), \quad |t \circ f_{j+1}^i|C_i - h \circ f_{j+1}^i|C_i| < \theta.
\]

By the proposition, \( t \circ f_{j+1}^i|C_i \equiv h \circ f_{j+1}^i|C_i \), and \( h \circ f_{j+1}^i|C_i \) is essential. Hence, C is an M-cycle on which k is essential.

The next result provides a characterization of essential maps from M into \( S^1 \) in terms of M-cycles.

**Theorem 2.** If k is a map from M into \( S^1 \), n is a positive integer, and h: \( X_{n+1} \to S^1 \) is a map such that \( |h \circ \pi_{n+1} - k| < \theta/2 \), then k is essential if and only if there is an M-cycle C, associated with h, on which k is essential.

**Proof.** Let k be a map from M into \( S^1 \). Let n be a positive integer and h: \( X_{n+1} \to S^1 \) a map such that \( |h \circ \pi_{n+1} - k| < \theta/2 \).

Suppose k is essential. By Theorem 1, \( h \circ f_{n+1}^j \) is essential for each \( j \geq n+1 \). Since \( X_{n+2} \) is a locally connected continuum, by Theorem 4 of [1, §56, X, p. 430], there is a s.c.c. D contained in \( X_{n+2} \) such that \( h \circ f_{n+1}^{n+2}|D \) is essential. Let D denote such a s.c.c. Then \( h|f_{n+1}^{n+2}(D) \) is also essential. Since the continuous image of a locally connected continuum is locally connected, there is a s.c.c. \( E = f_{n+1}^{n+2}(D) \) such that
h|E is essential. The sequence (E,D) has property p4. By a similar argument, for each integer \( j > 1 \), there is a s.c.c. \( H \) lying in \( X_{n+j} \) such that \( h \circ f_{n+1}^{n+j}|H \) is essential, and furthermore, there is a s.c.c. \( K \) lying in \( f_{n+1}^{n+j-1}(H) \) such that \( h \circ f_{n+1}^{n+j-1}|K \) is essential. Now, for each positive integer \( i \), \( X_{n+i} \) is a graph, and thus \( X_{n+i} \) contains only finitely many s.c.c.s. Therefore, for each positive integer \( j \), the set of all s.c.c.s \( K \) lying in \( X_{n+j} \) such that \( f_{n+1}^{n+j}|K \) is essential is finite. Using an argument analogous to that which shows the existence of an inverse limit on a sequence of finite spaces, each of which has the discrete topology, one deduces the existence of an infinite sequence of s.c.c.s having property p4 (e.g., Theorem 114 of [6]). Hence there is an M-cycle on which \( k \) is essential.

Now suppose \( C \) is an M-cycle associated with \( h \) on which \( k \) is essential. Let \( j \) be a positive integer. Since \( h \circ f_{n+1}^{n+j}|C_j \) is essential, so also is \( h \circ f_{n+1}^{n+j} \). By Theorem 1, \( k \) is essential.

The second section of this paper describes certain irreducibility properties that the inverse system \((X_i,f_i^{i+1})\) may satisfy. These properties will be related to the notion of an "irreducibly essential" map in the third section.

**Definition 2.** Suppose \( L \) is a compact subset of \( M \), \( n \) is a positive integer, and \( C \) is an M-cycle, with \( C_\perp \subset X_{n+1} \). Then \( L \) is said to be "projection-irreducible about the terms of \( C \)" (briefly, "\( L \) is irreducible with respect to \( C \)") provided that

1. for each \( i \), \( C_i \subset \pi_{n+i}(L) \), and
(2) for each compact, proper subset $T$ of $L$, there exists $j$ such that $C_j \notin \pi_{n+j}(T)$.

Theorem 3. If $n$ is a positive integer, $C$ is an $M$-cycle, $C_1 \subset X_{n+1}$, then there is a compact subset of $M$ which is irreducible with respect to $C$. Furthermore, each such point set is connected.

Proof. Let $n$ be a positive integer, $C$ an $M$-cycle, and $C_1 \subset X_{n+1}$. Let $H$ be the set of all compact subsets $K$ of $M$ such that, for each $i$, $C_i \subset \pi_{n+i}(K)$. Let $H$ be partially ordered by set inclusion: "$A \preceq B$" if and only if $A \subset B$. Let $L$ be a maximal, totally ordered subset of $H$. Let $Y$ be the common part of all elements of $L$.

Now, $Y$ is a member of $L$. For: Let $j$ be a positive integer, and $P$ a point of $C_j$. Let, for each $K$ in $L$, $g_K = \pi_{n+j}^{-1}(K)$. Suppose $A, B \in L$, and $A \preceq B$. Then $g_A = g_B|A$, whence $g_A^{-1}(P) \subset g_B^{-1}(P)$. We have $Q = \{g_K^{-1}(P): K \in L\}$ totally ordered by set inclusion, with $g_A^{-1}(P) \subset g_B^{-1}(P)$ whenever $A \preceq B$. Also, each member of $Q$ is a compact point set. Then $\bigcap_{K \in L} g_K^{-1}(P)$ is a compact point set; let $R$ denote it. Since $g_K^{-1}(P) \subset K$, for each $K$,

$$R \subset Y.$$ 

We have $C_j = \pi_{n+j}(Y)$ for each $j$, i.e., $Y \subset H$. Since $Y \subset K$ for each $K$ in $L$, and $L$ is maximal, $Y \subset L$. Also, since $L$ is maximal, no proper compact subset of $Y$ is in $H$ whence $Y$ is irreducible with respect to $C$.

Suppose $Z$ is compact and $Z$ is irreducible with respect to $C$. Suppose $Z$ is not connected. Let $Z = A \cup B$, the sum of 2 mutually exclusive, closed point sets. Let $j$ be a positive
integer such that, for each $i \geq 0$, $\pi_{n+j+i}(A)$ does not intersect $\pi_{n+j+i}(B)$.

Since $C_j$ is connected, either $C_j \subset \pi_{n+j}(A)$ or $C_j \subset \pi_{n+j}(B)$; assume $C_j \subset \pi_{n+j}(A)$. Let $i$ be a positive integer. Either $C_{j+i} \subset \pi_{n+j+i}(A)$ or $C_{j+i} \subset \pi_{n+j+i}(B)$; suppose $C_{j+i} \subset \pi_{n+j+i}(B)$. Then $f_{n+j+i}(C_{j+i}) \subset \pi_{n+j}(B)$. But $C_j \subset f_{n+j+i}(C_{j+i})$, a contradiction. We have $C_i \subset \pi_{n+i}(A)$ for each $i$, and $A$ is a compact, proper subset of $Z$, whence $Z$ is not irreducible with respect to $C$, a contradiction.

The next result asserts that we may assume that all $M$-cycles on which $k$ is essential have their first term in the same factor space and are associated with the same map.

**Lemma 3.** Suppose $k$ is a map from $M$ into $S^1$, and $D$ is an $M$-cycle on which $k$ is essential. Let $m$ and $n$ be non-negative integers, $m < n$, and $s$ and $t$ be maps from $X_{m+1}$ and $X_{n+1}$ respectively into $S^1$, with $D$ associated with $t$, and $s$ such that $|s \circ \pi_{m+1} - k| < \theta/2$. Then there is an $M$-cycle $E$ associated with $s$ such that $|s \circ \pi_{n+1} - t| < \theta$.

**Proof.** Let $k$ be a map from $M$ into $S^1$, and $D$ be an $M$-cycle on which $k$ is essential. Let $m < n$, $s$ and $t$ be maps from $X_{m+1}$ and $X_{n+1}$, respectively, into $S^1$. Let $D$ be associated with $t$ and let $|s \circ \pi_{m+1} - k| < \theta/2$. We have $|t \circ \pi_{n+1} - k| < \theta/2$, whence
\[ |s \circ \pi_{m+1} - t \circ \pi_{n+1}| = |s \circ f^{n+1}_{m+1} \circ \pi_{n+1} - t \circ \pi_{n+1}| < \delta, \]

and \[ |s \circ f^{n+1}_{m+1} - t| < \delta. \]

Since \( t \) is essential, so are \( s, f^{m+2}, f^{m+3}, \ldots, f^{n+1}_n \). The \( f^{n+1}_n \)-image of \( D_1 \) is a locally connected subcontinuum of \( X_n \).

Since \( s \circ f^{n}_{m+1} | f^{n+1}_n(D_1) \) is an essential map onto \( S^1 \), by Theorem 4 of [1, §56, X, p. 430], there is a simple closed curve \( L \) lying in \( f^{n+1}_n(D_1) \) such that \( s \circ f^{n}_{m+1} | L \) is essential; let \( H_1 \) denote such a s.c.c. Similarly, there is a s.c.c. \( K \) lying in \( f^{n-1}_n \)-image of \( H_1 \) such that \( s \circ f^{n-1}_{m+1} | K \) is essential; let \( H_2 \) denote such a s.c.c. Proceeding by induction, there is a sequence \( (H_1, H_2, \ldots, H_{n-m}) \) of simple closed curves, with \( H_i \subset X_{n+1-i}, H_{i+1} \subset f^{n+1-i}_{n-i}(H_i) \), and \( s \circ f^{n+1-i}_{m+1} | H_i \) essential for each \( i \). Let \( E \) denote the following sequence:

\[
E_j = \begin{cases} 
H_{n+1-m-j} & \text{if } 1 \leq j \leq n-m \\
D_{j-n+m} & \text{if } n-m < j
\end{cases}
\]

Then \( E \) is an M-cycle associated with \( s \) on which \( k \) is essential.

In the last section we prove the main theorem, which characterizes irreducibly essential maps from \( M \) onto \( S^1 \) in terms of M-cycles and the irreducibility condition discussed in the second section. The final result uses the main theorem to examine hereditary unicoherence in terms of inverse limit properties.

From definitions 1 and 2, and from Theorem 2, we have

**Theorem 4.** If \( k \) is a map from \( M \) onto \( S^1 \), then \( k \) is irreducibly essential if and only if (1) there is an M-cycle on which \( k \) is essential and (2) if \( W \) is an M-cycle on which \( k \) is essential, then \( M \) is irreducible with respect to \( W \).
Proof. Condition (1) is necessary and sufficient for \( k \) to be essential. For \( k \) to be inessential on every compact proper subset of \( M \), it is necessary and sufficient that \( k \) be inessential on every proper subcontinuum of \( M \). Suppose \( k \) is irreducibly essential. Let \( W \) be an \( M \)-cycle on which \( k \) is essential. Let \( H \) be a proper subcontinuum of \( M \). Then \( k|H \) is inessential. By Theorem 2, there is no \( H \)-cycle on which \( k|H \) is essential. Let \( j \) be a non-negative integer such that \( W \subseteq X_{j+1} \). By Lemma 2, if, for each \( i \), \( W_i \subseteq \pi_{j+1}(H) \), then \( W \) is a \( H \)-cycle on which \( k|H \) is essential, a contradiction. Hence \( M \) is irreducible with respect to \( W \).

Now suppose condition (2) holds. Let \( L \) be a proper subcontinuum of \( M \). Suppose \( n \) is a non-negative integer, and \( h \) is a map, \( h: X_{n+1} \to S^1 \), such that \( |k - h \circ \pi_{n+1}| < \theta/2 \). Suppose \( k|L \) is essential. By Theorem 2, let \( C \) be an \( L \)-cycle on which \( k|L \) is essential, with \( C \subseteq \pi_{n+1}(L) \). By Lemma 2, \( C \) is an \( M \)-cycle on which \( k \) is essential. But \( M \) is irreducible with respect to \( C \), a contradiction. Thus \( k|L \) is inessential, whence \( k \) is irreducibly essential.

**Theorem 5.** If \( n \) is a positive integer, and \( C \) is an \( M \)-cycle, \( C \subseteq X_{n+1} \), then \( M \) is irreducible with respect to \( C \) if and only if for each positive integer \( s \), and each number \( e > 0 \), there is a positive integer \( t > s \) such that, if \( x \in X_{n+s} \), then \( d_{n+s}(x,f_{n+s}(C_t)) < e \).

Proof. Let \( n \) be a positive integer, \( C \) an \( M \)-cycle, and \( C \subseteq X_{n+1} \). Suppose \( M \) is irreducible with respect to \( C \). Let \( s \) be a positive integer and \( e > 0 \). Suppose, by way of contradiction, that for every \( t > s \) there is a point \( x \in X_{n+s} \)
such that $d_{n+s}(x, f_{n+s}^n(C)) \geq e$.

Let $W$ be the following sequence: if $i$ is a positive integer, then

$$W_i = \{x \in X_{n+s}: d_{n+s}(x, f_{n+s}^{n+i}(C_{s+i})) \geq e\}.$$

Now, for each $i$, $W_i$ is closed in $X_{n+s}$. Also since

$$C_{s+i} \subseteq f_{n+s}^{n+i+1}(C_{s+i+1}) \text{, and}$$
$$f_{n+s}^{n+i}(C_{s+i}) \subseteq f_{n+s}^{n+i+1}(C_{s+i+1}),$$

for each $i$, we have $W_{i+1} \subseteq W_i$. Since each term of $W$ is compact, $\bigcap W_i$ is a point set; denote it by $Y$. If $x \in Y$, then for every $i$,

$$d_{n+s}(x, f_{n+s}^{n+i}(C_{s+i})) \geq e.$$

Let $q \in Y$, and let $0$ be the set of all points $x$ such that $d_{n+s}(x, q) < e/2$. By the triangle inequality, if $x \in 0$, then $d_{n+s}(x, f_{n+s}^{n+i}(C_{s+i})) > e/2$ for every $i$.

Now, $M - \pi_{n+s}^{-1}(0)$ is a closed, proper subset of $M$; denote it by $M'$. Let $j$ be a positive integer. Then

$$f_{n+s}^{n+j} \circ \pi_{n+s+j} = \pi_{n+s},$$
$$f_{n+s}^{n+s+j}(C_{s+j}) \subseteq X_{n+s} - 0 = \pi_{n+s}(M').$$

Suppose $C_{s+j} \neq \pi_{n+s+j}(M')$. Let $p \in C_{s+j}$, but $p \notin \pi_{n+s+j}(M')$. Then there is a point $p'$ in $M$ such that $\pi_{n+s+j}(p') = p$, but $p' \notin M'$. Hence $p' \in \pi_{n+s}^{-1}(0)$. We have $\pi_{n+s}(p') \in 0$. Also,

$$f_{n+s}^{n+s+j}(\pi_{n+s+j}(p')) \in 0, \text{ and } f_{n+s}^{n+s+j}(p) \in 0.$$ But $f_{n+s}^{n+s+j}(C_{s+j}) \subseteq X_{n+s} - 0$, a contradiction. Thus $C_{s+j} \subseteq \pi_{n+s+j}(M')$. Also, for $1 \leq p \leq s$, $C_p \subseteq f_{n+p}^{n+s}(C_p)$, and $C_s \subseteq \pi_{n+s}(M')$, whence $C_p \subseteq \pi_{n+p}(M') = f_{n+p}^{n+s} \circ \pi_{n+s}(M')$. Hence $C_i \subseteq \pi_{n+i}(M')$ for each $i$, and $M$ is not irreducible with respect to $C$, a contradiction.
Now suppose that for each positive integer $s$ and each $e > 0$, there is an integer $t > s$ such that if $x \in X_{n+s}$, then $d_{n+s}(x, f_{n+s}^t(C_t)) < e$. Suppose $M'$ is a compact, proper subset of $M$. Let $P \in M - M'$. Let $O$ be a sub-basis element of $M$, and $P \in O$, and $O \cap M' = \emptyset$. Let $q$ be a positive integer, $L$ an open set in $X_q$, and $O = \pi_q^{-1}(L)$. Then $(f_q^{n+q})^{-1}(L)$ is open in $X_{n+q}$, and $P_{n+q} \in \pi_{n+q}(O) = (f_q^{n+q})^{-1}(L)$, with $\pi_{n+q}(O) \cap \pi_{n+q}(M') = \emptyset$. Let $e > 0$ such that $B(P_{n+q}, e) \subset \pi_{n+q}(O)$. Let $t$ be an integer, $t > q$, such that $d_{n+q}^t(P_{n+q}, f_{n+q}^t(C_t)) < e$.

If $C_t \subset \pi_{n+t}(M')$, then $f_{n+q}^t(C_t) \subset \pi_{n+q}(M')$, contradicting $\pi_{n+q}(M') \cap \pi_{n+q}(O) = \emptyset$. Thus $M$ is irreducible with respect to $C$.

From Theorems 4 and 5, we have immediately

**Theorem 6.** If $k$ is a map from $M$ onto $S^1$, then $k$ is irreducibly essential if and only if (1) there is an $M$-cycle on which $k$ is essential, and (2) if $n$ is a positive integer, and $C$ is an $M$-cycle on which $k$ is essential, with $C_1 \subset X_{n+1}$, then for each positive integer $s$, and each number $e > 0$, there is an integer $t > s$ such that, if $x \in X_{n+s}$, then $d_{n+s}(x, f_{n+s}^t(C_t)) < e$.

**Theorem 7.** Suppose $H$ is a continuum. Then $H$ is hereditarily unicoherent if and only if there is no decomposable subcontinuum $H'$ of $H$ which is the domain space of an irreducibly essential map onto $S^1$.

**Proof.** Suppose $H$ is hereditarily unicoherent. Let $H'$ be a subcontinuum of $H$, and $g$ an irreducibly essential map from $H'$ onto $S^1$. Suppose $H'$ is decomposable. Let $H'$ be the
sum of two proper subcontinua $A$ and $B$. Then $g|A$ and $g|B$ are inessential. But $H'$ is unicoherent; thus $A \cap B$ is connected, and $g$ is inessential, a contradiction.

Now suppose each subcontinuum of $H$ which is the initial set of an irreducibly essential map onto the circle is indecomposable. Suppose $H$ is not hereditarily unicoherent. Let $K$ be a subcontinuum of $H$, and $K = A \cup B$, the sum of two proper subcontinua, such that $A \cap B$ is not connected. Let $A \cap B = C \cup D$, the sum of two mutually exclusive closed sets. Let $A'$ be a subcontinuum of $A$ irreducible between $C$ and $D$. Let $B'$ be a subcontinuum of $B$ irreducible between $A' \cap C$ and $A' \cap D$. By Urysohn's lemma, let $f$ be a map from $A'$ onto the interval $[0,\frac{1}{2}]$ such that $f(A' \cap C) = 0$, $f(A' \cap D) = \frac{1}{2}$, and the $f$-image of every other point of $A'$ is in the open interval $(0,\frac{1}{2})$. Similarly, let $g$ be a map from $B'$ onto $[\frac{1}{2},1]$ such that $g(B' \cap A' \cap D) = \frac{1}{2}$, $g(B' \cap A' \cap C) = 1$, and the $g$-image of every other point of $B'$ is in the open interval $(\frac{1}{2},1)$. Letting $\emptyset$ denote the wrapping function from the real line into the plane, $\emptyset(x) = e^{2\pi i x}$, we define a function $h$ from $A' \cup B'$ into $S^1$, as follows:

$$h(x) = \begin{cases} 
\emptyset(f(x)) & \text{if } x \in A' \\
\emptyset(g(x)) & \text{if } x \in B'
\end{cases}$$

Then $h$ is an irreducibly essential map, whence $A' \cup B'$ is indecomposable, a contradiction.

Using the inverse limit characterization of indecomposability due to D. P. Kuykendall [5], we obtain the following result.

**Corollary.** Suppose $M$ is the inverse limit of an inverse
system of one-dimensional polyhedra, \( M = \lim_{+}(X_i, f^{i+1}_i, \pi_i) \).

Then \( M \) is hereditarily unicoherent if and only if the following condition holds: if \( H \) is a subcontinuum of \( M \),

\[
H = \lim_{+}(\pi_1(H), f^{i+1}_1|\pi_1(H), \pi_1|H) = \lim(Y_i, g^{i+1}_i, \sigma_i),
\]

\( n \) is a non-negative integer, \( t \) is a map from \( Y_{n+1} \) into \( S^1 \), and \( (1) \) there is an \( H \)-cycle \( C \) associated with \( t \), and \( (2) \) \( H \) is irreducible with respect to \( D \) for each \( H \)-cycle \( D \) associated with \( t \), then \( H \) is indecomposable, i.e., if \( m \) is a positive integer, and \( e > 0 \), then there are a positive integer \( W \) and three points of \( Y_W \) such that if \( P \) and \( Q \) are two of them, and \( K \) is a subcontinuum of \( Y_W \) containing \( P \) and \( Q \), then \( d_m(x, g^W_m(K)) < e \), for each point \( x \) of \( Y_m \).

References


Texaco

Bellaire, Texas 77401