A NOTE ON THE PRODUCT OF
FRECHET SPACES

by

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1. Introduction

A space $X$ is said to be a Fréchet space if whenever $x \in \mathbb{A}$, there exist $x_n \in A$, $n = 1, 2, \ldots$, with $x_n \to x$. In general, Fréchet spaces behave very badly with respect to products. In fact, if $X$ and $Y$ are non-discrete Fréchet spaces and $X \times Y$ is Fréchet, then a theorem of Michael [5] implies that $X$ and $Y$ must have the following stronger property: if $x \in \bigcap_{n=1}^{\infty} A_n$, where $A_1 \supseteq A_2 \supseteq \cdots$, then there exists $x_n \in A_n$ with $x_n \to x$. Spaces satisfying this property are called countably bi-sequential spaces. We should add that even if $X$ and $Y$ are countably bi-sequential, this does not guarantee that $X \times Y$ is Fréchet (see [4] or [6]).

In a letter to the author, F. Galvin asked the following question: if $X_0, X_1, X_2, \ldots$ are such that $\prod_{i \leq n} X_i$ is Fréchet (equivalently, countably bi-sequential) for all $n \in \omega$, must $\prod_{i \in \omega} X_i$ be Fréchet (equivalently, countably bi-sequential)? Y. Tanaka [8, Problem 2.6] has asked the same question. In this paper, we construct, assuming Martin's Axiom (MA), a Fréchet space $X$ such that $X^n$ is Fréchet for all $n \in \omega$, but $X^\omega$ is not Fréchet. The space $X$ is countable, and has only one non-isolated point.

Before proceeding with the construction of the example, we would like to mention some related problems. Bi-sequential spaces [5] are closed under countable products, so the space $X$ we construct is a countable countably bi-sequential space.
which is not bi-sequential. Others (e.g., Galvin [2], Malyhin [4], Olson [6]) have constructed such spaces assuming various axioms of set theory, but no real example has been found. (There are uncountable real examples, e.g., an uncountable $\ell$-product of the unit interval.) A space $X$ is called a $w$-space if whenever $x \in \tilde{A}_n$, $n = 1, 2, \ldots$, there exists $x_n \in A_n$ with $x_n \to x$. These spaces were introduced by the author in [3], and defined in terms of an infinite game, but this characterization, due to P. L. Sharma [7], is much better. Clearly, every $w$-space is countably bi-sequential, and the difference between the two classes of spaces does not, on the surface, look very large. But the following question, also asked by Galvin, remains open: if $X^n$ is a $w$-space for all $n \in \omega$, must $X^\omega$ be a $w$-space (or a Fréchet space)? A counterexample to this question would be about as far as one could go in this direction. Call $X$ a $c^*$-space (terminology due to Sharma) if $X$ has countable tightness and every countable subset of $X$ is first countable. It is easy to see that if $X^n$ is a $c^*$-space for every $n \in \omega$, then $X$ is a $c^*$-space. No real example of a space which is a $w$-space but not a $c^*$-space has been found. However, Galvin [1] has constructed such spaces assuming MA.

2. Construction of the Example

Unless otherwise stated, we use the letters $m$, $n$, and $k$ to denote natural numbers. The example is based on a construction, by induction on the ordinals less than the continuum $c$, of a certain collection of almost-disjoint subsets of $\omega$. To get us past an uncountable stage $\alpha < c$, we need the
Lemma (MA). Let \( \{ I_\alpha \}_{\alpha < \kappa} \), \( \kappa < \mathfrak{c} \), be a collection of infinite almost-disjoint subsets of \( \omega \). Suppose \( A \subset \omega^n \times \omega^m \), and \( \{\alpha(0), \alpha(1), \ldots, \alpha(m-1)\} \subset \kappa \) are such that

(1) \( A \subset \omega^n \times \prod_{j<m} I_\alpha(j) \)

(2) \( A \cap \{ (\prod_{i<n} \omega \setminus E(i)) \times (\prod_{j<m} I_\alpha(j) \setminus \{F(j)\}) \} \neq \emptyset \) whenever \( E(i) \) is a finite union of the \( I_\alpha \)'s, together with a finite subset of \( \omega \), and \( F(j) \) is a finite subset of \( \omega \). Then there is a sequence \( \hat{x}_0, \hat{x}_1, \ldots \) of elements of \( A \) such that

(i) \( C(\hat{x}_i) \cap C(\hat{x}_j) = \emptyset \) whenever \( i \neq j \), where \( C(\hat{x}) \) is the set of coordinates of \( \hat{x} \);

(ii) if \( \alpha < \kappa \), then \( I_\alpha \cap \{ \pi_i(\hat{x}_j) : i < n, j \in \omega \} \) is finite, where \( \pi_i \) is the projection on the \( i \)th coordinate.

Proof. Let \( P = \{ (f,F) : f \subset A, F \subset \kappa, \text{with } f \text{ and } F \) finite \}. Define \( (f,F) < (g,G) \) if and only if

(a) \( f \subset g \) and \( F \subset G \);

(b) if \( \hat{y} \in g \setminus f \), then \( \hat{y} \) is an element of \( A \cap \{ (\prod_{i<n} \omega \setminus \bigcup_{\alpha \in F} I_\alpha) \times (\prod_{j<m} I_\alpha(j) \setminus \bigcup_{\hat{x} \in f} C(\hat{x})) \} \).

So defined, \( (P,<) \) satisfies the CCC because there are only countably many possible \( f \)'s, and \( (f,F) \) and \( (f,G) \) are bounded by \( (f,F \cup G) \). For each \( \alpha < \kappa \), and \( i \in \omega \) let \( X_{\alpha,i} = \{ (f,F) \in P : |f| > i \text{ and } \alpha \in F \} \). \( X_{\alpha,i} \) is a dense open set in \( (P,<) \), so by MA, there is a compatible family \( \{ (f_{\alpha,i}, F_{\alpha,i}) \in X_{\alpha,i} : \alpha < \kappa, i \in \omega \} \). Pick \( \hat{x}_0 \in f_{\alpha(0),i(0)} \). If \( \hat{x}_0, \hat{x}_1, \ldots, \hat{x}_{k-1} \) have been chosen, pick \( \hat{x}_k \in f_{\alpha(k),i(k) \setminus \bigcup_{j<k} f_{\alpha(j),i(j)}} \). We claim that \( \hat{x}_0, \hat{x}_1, \ldots \) is the desired sequence. If \( j < k \), then since \( \hat{x}_k \in f_{\alpha(k),i(k) \setminus f_{\alpha(j),i(j)}} \), and by the compatibility,
the conclusion of property (b) is satisfied with $\hat{y} = \hat{x}_k$ and $f = f_{a(j)}, i(j)$. Hence $C(\hat{x}_j) \cap C(\hat{x}_k) = \emptyset$, and so property (i) of the conclusion of the lemma is satisfied. Now let $a < k$. If $\hat{x}_k \notin f_{a',1}$, then the conclusion of (b) is satisfied with $\hat{y} = \hat{x}_k$ and $F = f_{a',1}$. Since $a \in f_{a',1}$, the first $n$ coordinates of $\hat{x}_k$ miss $I_a$. Thus (ii) is satisfied, and this completes the proof.

Theorem (MA). There is a countable Fréchet space $X$ such that $X^n$ is Fréchet for all $n \in \omega$, but $X^\omega$ is not Fréchet.

Proof. We will construct a countable space $X_k$ for each $k \in \omega$, so that $\prod_{k<n} X_k$ is Fréchet for all $n \in \omega$, but $\prod_{k \in \omega} X_k$ is not Fréchet. We can then take $X$ to be the free union of the $X_k$'s.

To this end, we will construct a sequence $\{\mathcal{S}_n\}_{n \in \omega}$ of collections of infinite subsets of $\omega$ such that $\bigcup_{n \in \omega} \mathcal{S}_n$ is a maximal almost-disjoint collection. We then take $X_k$ to be the space $\omega \cup \{\omega\}$ with the points of $\omega$ isolated, and a neighborhood of $\omega$ is $\omega \cup \{\omega\}$ minus a finite union of elements of $\bigcup_{n \in \omega} \mathcal{S}_j$. It is easy to see that, in the space $\prod_{k \in \omega} X_k$, the point $(\omega, \omega, \cdots) \in \text{Cl}\{(n, n, \cdots) : n \in \omega\}$, but no sequence of the type $\{(n_k, n_k, \cdots) : k \in \omega\}$ converges to $(\omega, \omega, \cdots)$. Thus $\prod_{k \in \omega} X_k$ is not a Fréchet space.

We need to construct the $\mathcal{S}_k$'s so that every finite product of the $X_k$'s is Fréchet. First construct $I_k(n), n \in \omega$, so that $\{I_k(n) : n \in \omega, k \in \omega\}$ is an almost-disjoint collection of infinite subsets of $\omega$, with the additional property that for each $k \in \omega$ and finite subset $F$ of $\omega$, there is $n \in \omega$ with $F \subset I_k(n)$.
For each $n \in \omega$, let $A_n = P(\omega^n)$, and let $A = \bigcup_{n \in \omega} A_n$. Let $A = \{A_\alpha : \alpha < c\}$ so that each element of $A$ appears $c$ times in the well-ordering. For each $n \in \omega$, define $\beta(n) = n$. Now suppose $I_n(a)$ and $\beta(n)$ have been defined for all $\alpha < \kappa$, where $\omega \leq \kappa < c$, and $k \in \omega$. Let $\mathcal{G}(\kappa) = \{I_n(\alpha) : \alpha < \kappa, k \in \omega\}$.

Let $\beta(\kappa)$ be the least ordinal $\beta$ such that $\beta > \beta(\alpha)$ whenever $\omega \leq \alpha < \kappa$, and such that $A_\beta \subset \omega^n$ satisfies the following two properties:

(i) there are a set $J \subset \{0,1,\ldots,n-1\} = n$, and $\{I_j : j \in J\} \subset \mathcal{G}(\kappa)$ so that $A_\beta \subset (\prod \omega) \times (\prod J)$;

(ii) $A_\beta \cap [(\prod \omega \setminus E(i)) \times (\prod J \setminus F(j))] \neq \emptyset$ whenever $i \in J \setminus \bigcup_m W$, where $W_m$ is infinite and $W_m \cap W_{m'} = \emptyset$ if $m \neq m'$. Express $\omega$ as $\omega = \bigcup_{m \in \omega} W_m$, and let $I_m(\kappa) = \{I_i(\kappa) : k \in W_m, i \in n \setminus J\}$. The inductive step is now complete.

Let $\mathcal{G}_k = \{I_k(\alpha) : \alpha < c\}$, and let $X_k$ be as defined earlier. We have already shown that $\prod X_k$ is not Fréchet. It remains to prove that $\prod X_k$ is Fréchet for each $n \in \omega$. To this end, suppose $A \subset \prod X_k$, and $x \in \overline{A} \setminus A$. We need to show there exists $x_n \in A$ with $x_n \to x$. We will prove this for the case $A \subset \omega^n$ and $x = (\omega,\omega,\ldots,\omega) = \omega^\omega$, the other cases being trivial or reducible to a case similar to this one.
Let $\mathcal{G} = \bigcup_{n} \mathcal{G}_{n}$. Suppose $A \cap (\prod_{n} \omega \setminus \mathcal{E}(i)) = \emptyset$, where $\mathcal{E}(i)$ is a finite union of elements of $\mathcal{G}$. Then $A \subseteq \bigcup_{n} (\omega \times \cdots \times \omega \times \mathcal{E}(i) \times \omega \times \cdots \times \omega)$, so there exists $j(0) < n$ and $I_{j(0)} \in \mathcal{G}$ so that $I_{j(0)} \subseteq E(j(0))$, and $\omega^{n} \in \text{Cl}(A(0))$, where $A(0) = A \cap \big[\omega \times \cdots \times \omega \times I_{j(0)} \times \omega \times \cdots \times \omega\big]$. Now suppose $A(0) \cap \big[\prod_{n}(\omega \setminus \mathcal{E}(i)' : i \in n \setminus \{j(0)\})\big] \times (I_{j(0)} \setminus D(j(0))) = \emptyset$, where $E(i)'$ is a finite union of elements of $\mathcal{G}$ and $D(j(0))$ is a finite subset of $\omega$. (We are using the subscript to indicate position in the product, in order to simplify notation.) Then there exists $j(1) \in n \setminus \{j(0)\}$ so that $\omega^{n} \in \text{Cl}(A(1))$, where $A(1) = A(0) \cap \big[\omega \times \cdots \times \omega \times I_{j(1)} \times \omega \times \cdots \times \omega \times I_{j(0)} \times \omega \times \cdots \times \omega\big] = A(0) \cap \prod_{n}(\omega : i \in n \setminus \{j(0), j(1)\})$. We continue the process until we have a set $J = \{j(0), \cdots, j(m)\}$ and $A(m) \subseteq (\prod_{n} \omega) \times \prod_{j \in J} I_{j}$ with $\omega^{n} \in \text{Cl}(A(m))$ and $A(m) \cap \big[\prod_{n}(\omega \setminus \mathcal{E}(i)) \times (\prod_{j \in J} \mathcal{F}(j))\big] = \emptyset$ whenever $E(i)$ is a finite union of elements of $\mathcal{G}$ and $D(j)$ is a finite subset of $\omega$.

Choose $\kappa_{0}$ large enough so that $\{I_{j} : j \in J\} \subseteq \mathcal{G}(\kappa_{0})$. Now $A(m) = A_{\beta}$ for $c \beta$'s, so choose $\beta_{0} > \sup\{\beta(a) : a < \kappa_{0}\}$ such that $A(m) = A_{\beta_{0}}$. Then for any $\kappa_{0} \leq \kappa < c$, it is true that $A_{\beta_{0}} \cap J$, and $\kappa$ satisfy (i) and (ii) in the above construction of the $\mathcal{G}_{\kappa}$'s. Thus $\beta_{0} = \beta(\kappa)$ for some $\kappa_{0} \leq \kappa < c$, and we have the sequence $x_{0}, x_{1}, \cdots$ in $A_{\beta(\kappa)}$ that we chose in the construction. It is easy to see from the definition of $X_{i}$ that the set $\{\pi_{i}(x) : k \in W_{n}\}$ converges to $\omega$ in $X_{i}$ for each $i \leq n$, and since $C(x_{j}) \cap C(x_{k}) = \emptyset$ for $j \neq k$, then $\{x_{k} : k \in W_{n}\}$ converges to $\omega^{n}$. This completes the proof.

Remark. We can get an example with only one non-isolated
point as follows: let Y be the space which is the free union X of the $X_k$'s, with the points "00" identified to a single point $\hat{0}$. Let $\pi: X + Y$ be the projection. Define a neighborhood of $\hat{0}$ to be of the form $\pi(U_1 U \cdots U U_n U X_{n+1} U X_{n+2} U \cdots)$, where $U_i$ is an open set in $X_i$ containing $00$.

References


Auburn University
Auburn, Alabama 36830