MAPPING THEOREMS FOR PLANE CONTINUA

by

CHARLES L. HAGOPIAN
In 1927 Kuratowski [12, p. 262] defined a continuum $M$ to be of type $\lambda$ if $M$ is irreducible and every indecomposable continuum in $M$ is a continuum of condensation. If a continuum $M$ is of type $\lambda$, then $M$ admits a monotone upper semi-continuous decomposition to an arc with the property that each element of the decomposition has void interior relative to $M$ [13, Theorem 3, p. 216].

In 1933 Knaster and Mazurkiewicz [8] defined a continuum $M$ to be $\lambda$-connected if for every pair $p,q$ of points of $M$, there exists a continuum of type $\lambda$ in $M$ that is irreducible between $p$ and $q$. They pointed out that $\lambda$-connectivity is a natural generalization of $\alpha$-connectivity (arcwise connectivity) and gave two examples to show that unlike $\alpha$-connectivity, $\lambda$-connectivity is not a continuous invariant. The domain in each of their examples is not planar.

Knaster and Mazurkiewicz [8, p. 90] raised the question of whether there exist counterexamples to the invariance of $\lambda$-connectivity under continuous transformations in the plane. In this paper I prove that if $M$ is a $\lambda$-connected plane continuum and $f$ is a continuous function of $M$ into the plane, then $f[M]$ is $\lambda$-connected.

The following intermediate property (weaker than $\alpha$-connectivity but stronger than $\lambda$-connectivity) is defined in the last section of [8].

A continuum $M$ is $\delta$-connected if for each pair $p,q$ of
points of $M$, there exists a hereditarily decomposable continuum in $M$ that is irreducible between $p$ and $q$. The closure of any ray in $E^3$ (Euclidean 3-space) that limits on a disk is a $\lambda$-connected continuum that is not $\delta$-connected. Every hereditarily unicoherent $\lambda$-connected continuum is $\delta$-connected. It follows from Theorem 2 of this paper that $\delta$-connectivity and $\lambda$-connectivity are equivalent properties for plane continua.

In 1972 I [1] proved that every $\delta$-connected nonseparating plane continuum has the fixed-point property. Krasinkiewicz gave another proof of this theorem in [9].

There exists a ray $P$ in $E^3$ such that $P$ limits on a disk and the closure of $P$ is a continuous image of the topologist's sine curve. Hence $\delta$-connectivity is not a continuous invariant. However, I [4] proved that if $M$ is a $\delta$-connected continuum and $f$ is a continuous function of $M$ into the plane, then $f[M]$ is $\delta$-connected.*

Unfortunately, I [1, 3, 4, 5, 6, and 7] was unaware of Knaster and Mazurkiewicz's article [8] and called $\delta$-connected continua $\lambda$-connected. In 1974 Krasinkiewicz [10, Theorem 3.2] proved that every hereditarily unicoherent continuum that is

*The proof of Theorem 3 of [4] can be simplified considerably by replacing line 30 of page 280 through line 22 of page 282 with the following:

"element of $V_1$ that joins $q_2$ to $a_1$, and (2) $q_2$ is the last point of $[y_1, q_1]$ that can be joined to $a_1$ by an element of $V_1$. Define $K_1 = [p_1, a_1] \cup L_1 \cup [q_2, q_1]$. Let $Z_1 = K_1$. Note that $Z_1$ is a continuum in $S^2 - G_1$ that contains $\{p_1, q_1\}$."

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a continuous image of a δ-connected continuum is hereditarily decomposable. Although Krasinkiewicz said he was following Knaster and Mazurkiewicz [8], he also called δ-connected continua λ-connected. The second example of Knaster and Mazurkiewicz [8] shows that Krasinkiewicz's theorem does not hold for λ-connected continua. In this example the product of the pseudo-arc and a circle is projected onto the pseudo-arc. In [11] Krasinkiewicz proved several other interesting theorems for δ-connected continua that do not hold for λ-connected continua.

Let M be a plane continuum. A subcontinuum L of M is a link in M if L is either the boundary of a complementary domain of M or the limit of a convergent sequence of complementary domains of M. The following characterization of δ-connected plane continua is established in [3, Theorem 2].

Theorem 1. A plane continuum M is δ-connected if and only if each link in M is hereditarily decomposable.

An indecomposable subcontinuum I of a continuum M is terminal in M if there exists a composant C of I such that each subcontinuum of M that meets both C and M - I contains I.

Theorem 2. If a plane continuum M is λ-connected, then M is δ-connected.

Proof. According to Theorem 1, it suffices to show that every link in M is hereditarily decomposable. Suppose there exists a link in M that contains an indecomposable continuum I. It follows from [2, Theorem 2] and [4, Theorem 1] that I
is terminal in M. Hence there exists a composant C of I such that each subcontinuum of M that meets C and M - I contains I. Let p and q be points of C and I - C, respectively.

Since M is \(\lambda\)-connected, there exists a continuum K of type \(\lambda\) in M that is irreducible between p and q. Since K is a decomposable continuum in M that meets C and I - C, K meets M - I. Therefore K contains I, and this contradicts the fact that K is a continuum of type \(\lambda\) irreducible between p and q. Hence every link in M is hereditarily decomposable.

**Theorem 3.** Every \(\lambda\)-connected plane continuum that does not separate the plane has the fixed-point property.

**Proof.** Since every \(\delta\)-connected nonseparating plane continuum has the fixed-point property [1], this theorem follows immediately from Theorem 2.

**Theorem 4.** A plane continuum M is \(\lambda\)-connected if and only if M cannot be mapped continuously onto Knaster's chainable indecomposable continuum with one endpoint.

**Proof.** This follows from [5, Theorem 2] and Theorem 2.

**Theorem 5.** If M is a \(\lambda\)-connected plane continuum and \(f\) is a continuous function of M into the plane, then \(f[M]\) is \(\lambda\)-connected.

**Proof.** By Theorem 2, M is \(\delta\)-connected. Hence \(f[M]\) is \(\delta\)-connected [4, Theorem 5]. Therefore \(f[M]\) is \(\lambda\)-connected.

Still unanswered is the following:

**Question.** Is every continuous image of every \(\lambda\)-connected plane continuum \(\lambda\)-connected?
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References

University of Houston
Houston, Texas 77004
and
California State University
Sacramento, California 95819