UNORDERED TYPES OF ULTRAFILTERS

by

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Suppose that $\kappa$ is a cardinal. If $U$ and $V$ are ultrafilters on $\kappa$ and $f: \kappa \times \kappa$ is a function, we say that $f(U) = V$ if $\{f(H) \mid H \in U\} = V$. We say that $V \leq U$ if there exists an $f$ with $f(U) = V$. We say that $U$ and $V$ are of the same type (or $U = V$) if both $V \leq U$ and $U \leq V$. This is an equivalence relation and $\leq$ then induces a partial order (called the Rudin-Keisler order $[1,3,4]$) on the types of ultrafilters in $\beta\kappa$ (the set of all ultrafilters on $\kappa$).

Throughout this paper, a set of ultrafilters on $\kappa$ is called unordered if its members are pairwise incompatible in the Rudin-Keisler order. Information about this partial order clearly has applications to the study of $\beta\kappa$ as the Stone-Čech compactification of the discrete space of cardinality $\kappa$ and to the construction of other counterexamples in topology. An absence of set-theoretic restrictions is especially important.

It has previously been shown [3] that there are $2^\kappa$ unordered types of ultrafilters on $\kappa$. It is the purpose of this paper to present a proof of S. Shelah that there are $2^{2^\kappa}$ unordered types of ultrafilters on $\kappa$.

The free set lemma of A. Hajnal [2] says that if $|X| = \alpha$ and $\beta < \alpha$ and $F: X \to \mathcal{P}(X)$ satisfies $x \notin F(x)$ and $|F(x)| < \beta$, for all $x \in X$, then there is a $Y \subset X$ with $x \notin F(y)$ and $y \notin F(x)$ for all $x$ and $y$ in $Y$ and $|Y| = \alpha$.

If $2^{2^\kappa} > (2^\kappa)^+$, then setting $X = \beta\kappa$, $\alpha = 2^{2^\kappa}$, $\beta = (2^\kappa)^+$
and \( F(U) = \{ V \in \beta_\kappa \mid V \leq U \} \) for all \( U \in \beta_\kappa \), then we have immediately from the free set lemma that there are \( 2^{2^\kappa} \) unordered types of ultrafilters on \( \kappa \). The following theorem thus completes the proof.

**Theorem (Shelah).** There are \((2^\kappa)^+\) unordered types of ultrafilters on \( \kappa \).

**Proof.** If \( \mathcal{G} \subseteq \mathcal{P}(\kappa) \), let \( \mathcal{G}' = \mathcal{G} \cup \{ \kappa - G \mid G \in \mathcal{G} \} \) and \( \mathcal{G}^* = \{ \cap K \mid K \subseteq \mathcal{G}, K \text{ is finite, and } G \in K \text{ implies } (\kappa - G) \notin K \} \) and \( G \neq \emptyset \}. \) If \( K = \emptyset \), then \( \cap K = \kappa \).

Let \( J \) be an independent family of subsets of \( \kappa \): i.e.,

1. \( J \subseteq \mathcal{P}(\kappa) \),
2. \( J = J' \),
3. no term of \( J^* \) is empty.

Choose \( J \) of cardinality \( 2^\kappa \).

Define \( \Sigma = \{ 2^\kappa \} \) if \( 2^\kappa \) is regular. Otherwise let \( \Sigma \) be a cofinal in \( 2^\kappa \) set of uncountable regular cardinals with no limit of members of \( \Sigma \) belonging to \( \Sigma \).

Our task would be relatively simple if \( 2^\kappa \) were regular. Since \( 2^\kappa \) may be singular the standard technique of partitioning \( 2^\kappa \) into \( \Sigma \) is necessary as is the defining of \( P_\gamma \) below for \( \gamma < 2^\kappa \) and the reindexing of \( \kappa^\kappa \) and \( J \) in the middle of our inductive construction. For infinite \( \gamma \), observe by induction that the cardinality of \( P_\gamma \) is \( |\gamma| \); only in retrospect is it clear that \( P_\gamma \) is precisely those subsets of \( \kappa \) which might have been used by the \( \gamma \)th stage of our induction.

Index \( \kappa^\kappa = \{ g_\gamma \mid \gamma < 2^\kappa \} \) and \( J = \{ F_\gamma \mid \gamma < 2^\kappa \} \).

We now define \( P_\gamma \subseteq \mathcal{P}(\kappa) \) for each \( \gamma < 2^\kappa \) by induction. Let \( Q_\gamma = \cup_{\delta < \gamma} P_\delta \) and \( R_\gamma = \cup_{\delta < \gamma} Q_\gamma \).

If \( T \in R_\gamma^* \), \( F \subseteq (Q_\gamma - R_\gamma) \), \( f = g_\delta \) for some \( \delta < \gamma \), and there is an \( S \subseteq (J - R_\gamma)^* \) such that \( S \neq (T \cap S) \subseteq f^{-1}(F) \),
then define $S(T,F,f) = S$ for some such $S$. Otherwise $S(T,F,f)$ is undefined.

Define $P_\gamma$ to be the set of all $X \subset \kappa$ such that at least one of the following:

1. $X \in \mathcal{Q}_\gamma^* \cup \mathcal{Q}_\gamma^* \cup \{F_\delta\} \cup \{\kappa - F_\delta\}$ where $\delta$ is minimal for $F_\delta \in (J - \mathcal{Q}_\gamma)$, or

2. $X = f_{\delta}^{-1}(Y)$ for some $\delta < \gamma$ and $Y \in \mathcal{Q}_\gamma$, or

3. $X = S(T,F,f)$ for some $T \in R_\gamma^*$, $F \in \mathcal{Q}_\gamma - R_\gamma$, and $f = g_\delta$ for some $\delta < \gamma$.

Reindex $J = \{G_\gamma | \gamma < 2^K\}$ in such a way that, if $\sigma \in \Sigma$, then $\{G_\gamma | \gamma < \sigma\} = J \cap P_\sigma$.

The construction. By induction for each $\alpha < (2^K)^+$ we construct an ultrafilter $U_\alpha$ on $\kappa$; we then prove that the $U_\alpha$'s are unordered.

So fix $\alpha < (2^K)^+$ and assume that $U_\beta$ has been defined for all $\beta < \alpha$. Index $\{\beta < \alpha\} = \{a_\gamma | \gamma < 2^K\}$. Then reindex $\{\beta < \alpha\} = \{a_\gamma | \gamma < 2^K\}$, $\kappa^K = \{f_\gamma | \gamma < 2^K\}$ and $\rho(\kappa) = \{T_\gamma | \gamma < 2^K\}$ in such a way that, if $\sigma \in \Sigma$, $f = g_\delta$ for some $\delta < \sigma$, $\beta = a_\rho$ for some $\rho < \sigma$, and $T \in P_\sigma$, then $\{\gamma < \sigma | a_\gamma = \beta, f_\gamma = f, T_\gamma = T\}$ is stationary in $\sigma$. Since there are $\sigma$ disjoint stationary subsets of $\sigma$, and $\{g_\delta | \delta < \sigma\}, \{a_\rho | \rho < \sigma\}$ and $P_\sigma$ all have cardinality at most $\sigma$, this is no problem.

For each $\gamma < 2^K$ we now inductively construct a filter $U_\alpha(\gamma)$; $U_\alpha$ will be an extension of $U_{\gamma < 2^K} U_\alpha(\gamma)$ to an ultrafilter.

So assume that $\gamma < 2^K$ and let $V_\alpha(\gamma) = U_{\delta < \gamma} U_\alpha(\delta)$ be given. Let $\sigma$ be the minimal member of $\Sigma$ greater than $\gamma$.

Define $Z_\gamma = \{Z \subset P_\sigma | V_\alpha(\gamma) \subset Z, Z - V_\alpha(\gamma) \text{ is finite, } Z\}$
is a filter, and no term of \((Z \cup (J - Z'))^*\) is empty).

Our induction hypothesis is that \(U_\alpha(\delta) \in Z_\delta\) for all \(\delta < \gamma\).

Define \(U_\alpha(\gamma) = V_\alpha(\gamma)\) unless for some limit \(\lambda\) we have one of the following cases.

**Case (0).** \(\gamma = \lambda + 1\), \(T_\lambda \in P_\sigma\), \(f_\lambda = g_\delta\) for some \(\delta < \sigma\), and there is an \(F \in ((P_\sigma \cap J) - V_\alpha(\gamma)^*)\) such that \(S(T_\lambda, F, f_\lambda)\) is defined. In this case let \(U_\alpha(\gamma) = \{\kappa - F\} \cup V_\alpha(\gamma)\) for some such \(F\).

**Case (1).** \(\gamma = \lambda + 2\). Let \(\delta\) be minimal for \(G_\delta \in (J - V_\alpha(\gamma)^*)\); let \(F\) be the one of \(G_\delta\) and \((\kappa - G_\delta)\) such that \(f^{-1}(F)\) does not belong to \(U_\beta_\lambda\). Define \(U_\alpha(\gamma) = V_\alpha(\gamma) \cup \{F\}\) in this case. Observe that this case assures us that \(f_\lambda(U_\beta_\lambda) \neq U_\alpha\) and that \(U_\alpha(\sigma) \not\supset P_\sigma \cap J\).

Let \(U_\alpha\) be an arbitrary extension of \(\{U_\alpha(\gamma) \mid \gamma < 2^K\}\) to an ultrafilter. It remains to prove that \(\{U_\alpha \mid \alpha < (2^K)^+\}\) are unordered; (I) and (II) below complete this proof.

Assume \(\beta < \alpha < (2^K)^+\) and \(f \in K^K\). There are \(\mu\) and \(\eta\) in \(2^K\) and \(\sigma \in \Sigma\) such that \(f = g_\mu\) and \(\beta = \alpha \eta\), \(\mu < \sigma\) and \(\eta < \sigma\).

Let \(\Lambda = \{\lambda < \sigma \mid \lambda\) is a limit and \(\beta_\lambda = \beta\) and \(f_\lambda = f\) (in the \(\alpha\) indexing)\}.

(I) \(f(U_\beta) \neq U_\alpha\).
Proof. By our indexing there is a \( \lambda \in A \) and by case (2) \( f(U_\beta) \neq U_\alpha \).

\((\text{II})\) \( f(U_\alpha) \neq U_\beta \).

Proof. For \( T \in (U_\alpha \cap P_\gamma)^* \), let
\[
\Delta_T = \{ \delta < \sigma \mid S(T,F,f) \text{ is defined for some } F \in ((J - P_\sigma) - P_\delta) \}.
\]

Case (a). There is a \( T \) with \( \Delta_T = \sigma \).

Choose \( \lambda \in \Lambda \) with \( T_\lambda = T \). There is a \( \gamma \in \sigma \) with \( U_\alpha(\lambda) \subseteq P_\gamma \).

Choose a limit \( \lambda' < \sigma \) in the \( \beta \) indexing with \( f = f_\lambda' \), and \( T = T_\lambda' \), and \( (J - P_\gamma) \subseteq V_\beta(\lambda')' \); by our indexing and case (2) this is possible. Since there is a \( \delta < \sigma \) with \( V_\beta(\lambda')' \subseteq P_\delta \) and \( \Delta_T = \sigma \), there is an \( F \in ((P_\sigma \cap J) - V_\beta(\lambda')') \) such that \( S(T,F,f) \) is defined. Thus, by case (1), there is a \((\kappa - F) \in U_\beta \) for some such \( F \). Since \( F \not\in V_\beta(\lambda')' \Rightarrow (P_\gamma \cap J) \), \( F \in Q_\rho - R_\rho \) for some \( \rho > (\gamma + 1) \). Thus \( S = S(T,F,f) \in (J - R_\rho)' \subseteq (J - P_\gamma)' \subseteq (J - U_\alpha(\lambda))' \); also \( S \in P_\sigma \). Thus by our inductive hypotheses, \( Z = (V_\alpha(\lambda) \cup S) \in Z_\Lambda \). Since \( T \in V_\alpha(\lambda) \), \( Y = (T \cap S) \in Z_\Lambda ^* \). Since \( Y \subseteq f^{-1}(F) \) and \( (\kappa - F) \in U_\beta \), by case (0), we chose such a \( Z = U_\alpha(\lambda) \), hence such an \((\kappa - F) \in U_\alpha \). So \((\kappa - F) \in U_\beta \) implies \( U_\alpha \neq U_\beta \).

Case (b). \( \Delta_T < \sigma \) for all \( T \).

For each \( \delta < \sigma \) choose \( \delta^* < \sigma \) such that, for all \( T \in U_\alpha(\delta) \), \( \Delta_T \subseteq \delta^* \), \( ((P_\delta \cap J) \subseteq U_\alpha(\delta^*)' \) and \( U_\alpha(\delta)' \subseteq P_\delta^* \). Choose \( \lambda \in \Lambda \) such that \( \gamma < \lambda \) implies \( \gamma^* < \lambda \). Then choose \( F \in (P_\lambda \cap J) - Q_\lambda^* \) and let \( F \) be the one of \( F \) and \( (\kappa - F) \) which belongs to \( U_\beta \).
If \( \{f^{-1}(\kappa - F)\} \cup \mathcal{V}_\alpha(\lambda) \in Z_\lambda \), then, by case (0)
\[
f(U_\alpha) \neq U_B.
\]

If \( \{f^{-1}(\kappa - F)\} \cup \mathcal{V}_\alpha(\lambda) \notin Z_\lambda \), then there is an
\[
S \in (J - \mathcal{V}_\alpha(\lambda))^* \quad \text{and} \quad T \in \mathcal{V}_\alpha(\lambda)^* \quad \text{such that} \quad \emptyset \neq (S \cap T) \subset f^{-1}(F).
\]
Since, for all \( \delta < \lambda \), \( (P_\delta \cap J) \subset U_\alpha(\delta)^* \) and
\[
U_\alpha(\delta)^* \subset P_\delta^*, \quad (Q_\lambda \cap J) \subset \mathcal{V}_\alpha(\lambda)^* \quad \text{and} \quad \mathcal{V}_\alpha(\lambda) \subset Q_\lambda.
\]
Thus
\[
F \in (Q_{\lambda+1} - R_{\lambda+1}), \quad S \in (J - R_{\lambda+1}^*) \quad \text{and} \quad T \in R_{\lambda+1}^*.
\]
Hence
\[
S(T,F,f) \quad \text{is defined. But} \quad T \notin U_\alpha(\delta) \quad \text{for some} \quad \delta < \lambda, \quad \delta^* < \lambda, \quad \text{and} \quad \Delta_T \subset \delta^*.
\]
Since \( F \notin Q_\lambda \), this is a contradiction of the definition of \( \Delta_T \).

Bibliography


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