SOME PROPERTIES OF WHITNEY CONTINUA

by

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1. Introduction

A continuum is a compact connected metric space. The letter X will always denote a continuum with metric d, and C(X) is the hyperspace of nonempty subcontinua of X metrized by the Hausdorff metric H. For basic facts about hyperspaces, see [12]. If A ∈ C(X), then C(A) = {Y ∈ C(X) | Y ⊆ A} and \( \hat{A} = \{\{a\} | a ∈ A\} \). A continuous map \( \mu : C(X) → R \) is called a Whitney map if it satisfies: (1) \( \mu(\{x\}) = 0 \) for each \( x ∈ X \), and (2) if \( A \subseteq B \) and \( A \neq B \), then \( \mu(a) < \mu(B) \). Whitney [16] has shown that such maps always exist. Throughout this paper, \( \mu \) will stand for an arbitrary Whitney map on C(X).

It is known [4] that \( \mu \) is monotone, i.e., \( \mu^{-1}(t) \) is a sub-continuum of C(X) for each t. The continua \( \mu^{-1}(t) \) are called the Whitney continua. Notice that if \( A ∈ C(X) \), then \( C(A) \cap \mu^{-1}(t) \) is a continuum since it is a Whitney continuum in C(A).

A topological property P is said to be a Whitney property provided that whenever a continuum X has property P, so does \( \mu^{-1}(t) \) for each Whitney map \( \mu \) for C(X) and each t with \( 0 < t < \mu(X) \). Whitney properties were investigated by several authors (see [8], [14], [15], and, for a summary of results, see [12]). Nadler [12] defines a topological

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property $P$ to be a *strong Whitney-reversible property* (resp., *Whitney-reversible property*) provided that whenever $X$ is a continuum such that $\mu^{-1}(t)$ has property $P$ for some Whitney map (resp., all Whitney maps) $\mu$ for $C(X)$, and all $t$ with $0 < t < \mu(X)$, then $X$ has property $P$. Nadler ([12], [13]) has shown that some topological properties are Whitney-reversible and he asked [12, (14.57)] if certain other properties are Whitney-reversible. In section 2 we show that hereditary decomposability, hereditary arcwise connectedness, and $C^*$-smoothness are strong Whitney-reversible properties.

In section 3 we study the relation between convexity of the Whitney continua and that of the underlying continuum.

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2. Whitney-Reversible Properties

A continuum is said to be *decomposable* provided that it is the union of two proper subcontinua. It is said to be *indecomposable* provided that it is not decomposable. A property $P$ of a continuum $X$ is said to be *hereditary* provided that each subcontinuum of $X$ has $P$. We will denote by $\sigma$ the union function $\sigma: C(C(X)) \to C(X)$ defined by $\sigma(\alpha) = \bigcup \{A | A \in \alpha\}$, and by $\widehat{i}$ the function $\widehat{i}: C(X) \to C(C(X))$ defined by $\widehat{i}(A) = \hat{A}$. It is known that $\sigma$ is continuous [6], and that $\widehat{i}$ is an isometry [12, (16.6)].

It is known [12, p. 413] that indecomposability is not a Whitney property. However, this result shows that indecomposability of $X$ is reflected in $\mu^{-1}(t)$. 
2.1. Theorem. Let $X$ be an indecomposable continuum. Let $\mu$ be a Whitney map for $C(X)$. Then for each $t \in (0, \mu(X))$ there exists an indecomposable continuum $\beta_t \subseteq \mu^{-1}(t)$ such that $\sigma(\beta_t) = X$.

Proof. Let $t \in (0, \mu(X))$ be fixed. It follows by the continuity of the union function $\sigma$ and Brouwer's reduction theorem that $\mu^{-1}(t)$ contains a continuum $\beta_t$ which is irreducible with respect to the property that $\sigma(\beta_t) = X$. We show that $\beta_t$ is indecomposable. For, if $\beta_t$ were the union of two proper subcontinua $\beta_1$ and $\beta_2$, then $\sigma(\beta_1)$ and $\sigma(\beta_2)$ would be proper subcontinua of $X$ such that $X = \sigma(\beta_1) \cup \sigma(\beta_2)$. This contradicts the fact that $X$ is indecomposable.

It is known (see [12, p. 454]) that decomposability is not a Whitney property.

2.2. Theorem. Assume there is a sequence $\{t_n\}_{n \in \omega}$ such that $t_n \to 0$ as $n \to \omega$, and $\mu^{-1}(t_n)$ is hereditarily decomposable for each $n = 1, 2, 3, \ldots$, then $X$ is hereditarily decomposable. Hence, hereditary decomposability is a strong Whitney-reversible property.

Proof. Suppose on the contrary that $X$ contains an indecomposable continuum $Y$. It follows easily from the continuity of $\mu$, and the hypothesis of the theorem that there exists $t_o \in \{t_n | n \in \omega\}$ such that $C(Y) \cap \mu^{-1}(t_o)$ is a non-degenerate subcontinuum of $\mu^{-1}(t_o)$. Then, by 2.1, there exists an indecomposable continuum $\beta \subseteq C(Y) \cap \mu^{-1}(t_o)$. This contradicts the fact that $\mu^{-1}(t_o)$ is hereditarily decomposable.
The result just proved answers one of the questions in [12, (14.57)].

A continuum $X$ is unicoherent provided that $A \cap B$ is connected whenever $A$ and $B$ are subcontinua of $X$ such that $A \cup B = X$. A triod is a continuum $M$ which contains a subcontinuum $N$ such that the complement of $N$ in $M$ is the union of three nonempty mutually separated sets. A continuum is a-triodic provided it contains no triod. A continuum $X$ is chainable provided that for each $\varepsilon > 0$, there exists a continuous function $f : X \rightarrow \mathbb{R}$ such that $\text{diam}(f^{-1}(r)) < \varepsilon$ for each $r \in f(X)$.

Nadler has proved the following result (see [12, (14.46), (14.49-51)].

2.3. Theorem [Nadler]. Assume there is a sequence $\{t_n\}_{n \in \omega}$ such that $t_n \to 0$ as $n \to \infty$ and $\mu^{-1}(t_n)$ is unicoherent (or, respectively, a-triodic, an arc, a circle), then $X$ is unicoherent (or, respectively, a-triodic, an arc, a circle).

The following two results provide partial answers to the question of whether chainability is a Whitney-reversible property.

2.4. Theorem. Assume there is a sequence $\{t_n\}_{n \in \omega}$ such that $t_n \to 0$ as $n \to \infty$ and $\mu^{-1}(t_n)$ is an hereditarily decomposable chainable continuum for each $n = 1, 2, 3, \cdots$, then $X$ is an hereditarily decomposable chainable continuum.

Proof. It follows by 2.2 that $X$ is hereditarily decomposable. Since a chainable continuum is hereditarily unicoherent and a-triodic, it follows by 2.3 that $X$ is hereditarily
unicoherent and a-triodic. Bing [2, Theorem 11] has proved that an hereditarily decomposable continuum is chainable if and only if it is a-triodic and hereditarily unicoherent.

A continuum X is said to have property \([K]\) provided that for each \(\varepsilon > 0\), there exists \(\delta > 0\) such that if \(a, b \in X\), \(d(a, b) < \delta\), and \(a \in A \in C(X)\), then there exists \(B \in C(X)\) such that \(b \in B\), and \(H(A, B) < \varepsilon\). It is known [6] that if \(X\) has property \([K]\), then the function \(F : X \times \{0, \mu(X)\} \to C(C(X))\) defined by \(F(x, t) = \{A \in \mu^{-1}(t) \mid x \in A\}\) is continuous.

2.5. Theorem. Let \(X\) be a continuum which has property \([K]\). Assume there is a sequence \(\{t_n\}_{n \in \omega}\) such that \(t_n \to 0\) as \(n \to \infty\), and \(\mu^{-1}(t_n)\) is chainable for each \(n = 1, 2, \ldots\), then \(X\) is chainable.

Proof. Let \(\varepsilon > 0\) be given. By the continuity of \(\mu\), and the hypothesis of the theorem, there exists \(t_0 \in \{t_n \mid n \in \omega\}\) such that \(diam(M) < \varepsilon/2\) for each \(M \in \mu^{-1}(t_0)\). Since \(\mu^{-1}(t_0)\) is chainable, there exists a continuous map \(g : \mu^{-1}(t_0) \to [0,1]\) such that \(diam(g^{-1}(r)) < \varepsilon/2\) for each \(r \in [0,1]\). Define \(f : X \to [0,1]\) by \(f(x) = centre(g(F(x, t_0)))\). Since \(X\) has property \([K]\), \(f\) is continuous. Let \(r \in f(X)\), and let \(a, b \in f^{-1}(r)\). Then there exist \(A \in F(a, t_0)\) and \(B \in F(b, t_0)\) such that \(r = g(A) = g(B)\). Since \(g\) is an \(\varepsilon/2\)-map, \(H(A, B) < \varepsilon/2\). Thus, \(d(a, b) < \varepsilon\). This shows that \(f\) is an \(\varepsilon\)-map. Hence, \(X\) is chainable.

It is known (see [12, (14.48)]) that arcwise connectedness is not a Whitney-reversible property. Let us note the following:
2.6. Theorem. Assume that $\mu^{-1}(t)$ is hereditarily ar­
wise connected for each $t \in (0, \mu(X))$, then $X$ is an arc or a
circle. Hence, hereditary arcwise connectedness is a strong
Whitney-reversible property.

Proof. It is known [9, p. 212] that each arcwise con­
nected continuum is decomposable. Thus, each $\mu^{-1}(t)$ is hered­
itarily decomposable for each $t \in (0, \mu(X))$. Then, by 2.2,
$X$ is hereditarily decomposable. It follows by [8, (3.3)] that
$X$ is a-triodic. Now, we show that $C(X) \setminus \{E\}$ is arcwise con­
nected for each proper subcontinuum $E$ of $X$. Let $E$ be an
arbitrary but fixed subcontinuum of $X$. We may assume that
$E$ is non-degenerate. To prove that $C(X) \setminus \{E\}$ is arcwise con­
nected, it suffices from the arc structure of $C(X)$ to show
that if $A$ is a proper subcontinuum of $E$, then $A$ and $X$ can be
joined by an arc in $C(X) \setminus \{E\}$. Let $t > 0$ be chosen such that
$\mu(A) \leq t < \mu(E)$. Let $B \in \mu^{-1}(t)$ such that $A \subseteq B$, and let $\alpha_1$
be an order arc from $A$ to $B$ (see [12]). Let $g \in X \setminus E$, and let
$G \in \mu^{-1}(t)$ such that $g \in G$. Since $\mu^{-1}(t)$ is arcwise con­
nected, there exists an arc $\alpha_2$ joining $B$ and $G$ in $\mu^{-1}(t)$.
Let $\alpha_3$ be an order arc from $G$ to $X$. It follows that $\alpha_1 \cup \alpha_2$
$\cup \alpha_3$ is an arc joining $A$ and $X$ in $C(X) \setminus \{E\}$. This shows that
$C(X) \setminus \{E\}$ is arcwise connected. Since $X$ is a-triodic and
hereditarily decomposable, it follows by [12, (11.16)] that
$X$ is chainable or circle-like.

If $X$ is chainable, then since the property of being a
chainable continuum is a Whitney property [7], each $\mu^{-1}(t)$
is chainable, $0 < t < \mu(X)$. Since each arcwise connected
chainable continuum is an arc, each $\mu^{-1}(t)$ is an arc. Then,
by 2.3, $X$ is an arc. On the other hand, if $X$ is circle-like
and not chainable (i.e., proper circle-like), then since the property of being a proper circle-like continuum is a Whitney property [7], each $\mu^{-1}(t)$ is a proper circle-like continuum, $0 < t < \mu(X)$. Thus, each $\mu^{-1}(t)$ is an hereditarily arcwise connected circle-like continuum. By [11, Theorem 6], each $\mu^{-1}(t)$ is a circle. Thus, by 2.3, $X$ is a circle.

A continuum $X$ is said to be $C^*$-smooth provided that the function $C^*: \mathcal{C}(X) \to \mathcal{C}(\mathcal{C}(X))$ defined by $C^*(A) = C(A)$ is continuous [12, (15.5)].

We denote by $H^2$ the Hausdorff metric on $\mathcal{C}(\mathcal{C}(X))$ corresponding to $H$ as a metric on $\mathcal{C}(X)$, and by $H^3$ the Hausdorff metric on $\mathcal{C}(\mathcal{C}(\mathcal{C}(X)))$ corresponding to $H^2$ as a metric on $\mathcal{C}(\mathcal{C}(X))$.

2.7. Lemma. For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $t < \delta$, $A$ is any subcontinuum of $X$, $\mu^{-1}(t)$ is hereditarily unicoherent, and $\beta$ is any subcontinuum of $\mu^{-1}(t)$ such that $\sigma(\beta) = A$, then $H^3(\mathcal{C}(\hat{A}), \mathcal{C}(\beta)) \leq \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. By the continuity of $\mu$ and the compactness of $\mathcal{C}(X)$, there exists $\delta > 0$ such that if $0 < t < \delta$, and $M \in \mu^{-1}(t)$, then diam($M$) < $\varepsilon$. Assume that $\mu^{-1}(t)$ is hereditarily unicoherent for some $t < \delta$. Let $A \in \mathcal{C}(X)$, and let $\beta \in \mathcal{C}(\mu^{-1}(t))$ such that $\sigma(\beta) = A$. Now,

$$H^3(\mathcal{C}(\hat{A}), \mathcal{C}(\beta)) = \max\{ \sup_{M \in \mathcal{C}(\beta)} \inf_{N \in \mathcal{C}(\hat{A})} H^2(M, N), \sup_{N \in \mathcal{C}(\hat{A})} \inf_{M \in \mathcal{C}(\beta)} H^2(M, N) \}.$$

If $M \in \mathcal{C}(\beta)$, let $N = (\hat{\delta}M)$. Then it is easy to see that $H^2(M, N) < \varepsilon$. On the other hand, if $N \in \mathcal{C}(\hat{A})$, let $X(\sigma(\mathcal{N}), \mu, t) = \{ G \in \mu^{-1}(t) | G \cap \sigma(\mathcal{N}) \neq \emptyset \}$. Then, by [8, (3.2)], $X(\sigma(\mathcal{N}), \mu, t)$ is a subcontinuum of $\mu^{-1}(t)$. Let
M = X(σ(N), μ(t)) ∩ β. Since μ⁻¹(t) is hereditarily unicoherent, M is a continuum, and once again H²(M, N) < ε. This shows that H³(C(β), C(β)) < ε.

2.8. Example. The following example shows that the assumption that μ⁻¹(t) is hereditarily unicoherent cannot be dropped from Lemma 2.7. Let X be the unit circle, and let μ be any Whitney map for C(X). Note that μ⁻¹(t) is a circle for each t ∈ (0, μ(X)) [7]. Let ε = 1/10. We show that for any t ∈ (0, μ(X)), there exists a subcontinuum S ~ μ⁻¹(t) such that a(S) = X, and H²(C(μ⁻¹(t)), C(S)) > 1/10. Let t ∈ (0, μ(X)) be arbitrary but fixed. It suffices to assume that diam(M) < 1/4 for each M ∈ μ⁻¹(t). Let λ > 0 such that diam(M) > λ for each M ∈ μ⁻¹(t). Let S be an open interval of X of length ~, and let X₁ = X\S. Let β = {M ∈ μ⁻¹(t) | M ∩ X₁ ≠ ∅}. Then β is a subcontinuum of μ⁻¹(t) such that σ(β) = X. Let N be the arc of X of length = 1 which contains S in its middle. It is easy to see that H²(N, γ) > 1/10 for each subcontinuum γ ⊂ β, and consequently H³(C(β), C(β)) > 1/10.

2.9. Theorem. Assume there is a sequence {t_n}ₙ∈ω such that t_n → 0 as n → ∞, and μ⁻¹(t_n) is C*-smooth for each n = 1, 2, ···. Then, X is C*-smooth. Hence, C*-smoothness is a strong Whitney-reversible property.

Proof. Let {Aₙ}ₙ∈ω be a sequence in C(X) such that lim Aₙ = A. To prove that X is C*-smooth, it suffices to show that if {C(Aₙ)}ₖ∈ω is any convergent subsequence of the sequence {C(Aₙ)}ₙ∈ω, then lim C(Aₙ) = C(A). We may assume that A is non-degenerate. Let A = lim C(Aₙ)ₖ∈ω such that Aₙ → A as n → ∞.
be arbitrary. Let $\delta > 0$ be chosen as in Lemma 2.7 with $\varepsilon$ replaced by $\varepsilon/3$. Let $t \in \{t_n | n \in \omega\}$ such that $t < \delta$, and such that $C(A) \cap \mu^{-1}(t)$ is a non-degenerate continuum. Then, by \cite{5, (2.1)}), \( \lim_{j \to \infty} (C(A_n) \cap \mu^{-1}(t)) = A \cap \mu^{-1}(t) \). Since $\mu^{-1}(t)$ is C*-smooth, there exists a natural number $N$ such that for each $j \geq N$,

$$H^3(C(C(A_n) \cap \mu^{-1}(t)), C(A) \cap \mu^{-1}(t)) < \varepsilon/3.$$  \hfill (1)

We may assume that for each $j \geq N$, $\sigma(C(A_n) \cap \mu^{-1}(t)) = A_n$. Since each C*-smooth continuum is hereditarily unicoherent \cite{5}, it follows by 2.7 that

$$H^3(C(A) \cap \mu^{-1}(t)), C(A) \cap \mu^{-1}(t)) < \varepsilon/3.$$  \hfill (2)

Since the union function $\sigma$ is continuous, $A = \sigma(A \cap \mu^{-1}(t))$. Hence, by 2.7

$$H^3(C(A) \cap \mu^{-1}(t)), C(A) \cap \mu^{-1}(t)) < \varepsilon/3.$$  \hfill (3)

It follows from (1), (2), and (3) and the triangle inequality that $H^3(C(A) \cap \mu^{-1}(t), C(A)) < \varepsilon$ for each $j \geq N$. Since for each $M \in C(A)$, and each $N \in C(A)$, $H^2(M, N) = H(\sigma(M), \sigma(N))$, it follows that $H^2(C(A) \cap \mu^{-1}(t)) < \varepsilon$ for each $j \geq N$. Consequently,

$$H^2(C(A) \cap \mu^{-1}(t)) < \varepsilon.$$  Since $\varepsilon$ is arbitrary, $A = C(A)$ and the proof is complete.

2.10. Remark. In contrast with 2.9, let us show that C*-smoothness is not a Whitney property. By \cite{12, (15.11)} a locally connected continuum is C*-smooth if and only if it is a dendrite. Let $X$ be a simple triod (a continuum homeomorphic to $\{(0,y) \in \mathbb{R}^2 | 0 \leq y \leq 1\} \cup \{(x,1) \in \mathbb{R}^2 | -1 \leq x \leq 1\}$). Then $X$ is C*-smooth. It follows by \cite{12, (14.9)} that $\mu^{-1}(t)$ is a
locally connected continuum for each \( t \in (0, \mu(X)) \). It is easy to see that \( \mu^{-1}(t) \) contains a 2-cell for each \( t \in (0, \mu(X)) \), and, therefore, \( \mu^{-1}(t) \) is not C*-smooth.

### 3. Convexity

A continuum \( X \) is said to be convex provided that for each pair of points \( x, y \in X \), there exists a point \( z \in X \setminus \{x, y\} \) such that \( d(x,z) + d(z,y) = d(x,y) \). It is known that if \( X \) is convex, then each pair of points of \( X \) can be joined by a segment in \( X \).

Let us note the following theorem for which we will show the converse is false.

#### 3.1. Theorem

Assume there is a sequence \( \{t_n\}_{n \in \omega} \) such that \( t_n \to 0 \) as \( n \to \infty \), and \( \mu^{-1}(t_n) \) is convex (with respect to the Hausdorff metric), then \( X \) is convex (with respect to the original metric \( d \) on \( X \)).

**Proof.** Since \( \mu \) is an open map [4], and \( \lim_{n \to \infty} t_n = 0 \), \( \lim_{n \to \infty} \mu^{-1}(t_n) = \hat{X} \). Since each \( \mu^{-1}(t_n) \) is convex, it follows by [3, (4.8)] that \( \hat{X} \) is convex, and consequently \( X \) is convex.

#### 3.2. Example

The following is an example of a convex arc \( X \), and a Whitney map \( \mu \) for \( C(X) \), such that \( \mu^{-1}(t) \) is not convex for any \( t \in (0,1] \). Let \( X = [0,3] \) with the Euclidean metric. Define a homeomorphism \( f: [0,3] \to [0,6] \) as follows:

\[
f(x) = \begin{cases} 
  x, & \text{if } x \in [0,1] \\
  x^2, & \text{if } x \in [1,2] \\
  2x, & \text{if } x \in [2,3].
\end{cases}
\]
Define \( \mu : C(X) \rightarrow [0, \infty) \) by \( \mu ([a,b]) = f(b) - f(a) \). Then, \( \mu \) is a Whitney map for \( C(X) \). We show that \( \mu^{-1}(t) \) is not convex.

Let \( t \in (0,1] \) be fixed. Let \( A = [0,t], B = [3-t/2,3], \) and \( D = [1,1+t] \). Then \( A, B \) and \( D \in \mu^{-1}(t) \). It is known that \( \mu^{-1}(t) \) is an arc [7]. Note that \( A \) and \( B \) are the end points of \( \mu^{-1}(t) \). It is easy to see that \( H(A,D) = 1, H(D,B) = 3 - \sqrt{1+t}, \) and \( H(A,B) = 3 - t/2 \). Thus, \( H(A,B) \neq H(A,D) + H(D,B) \).

This shows that \( \mu^{-1}(t) \) is not convex.

3.3. Remark. It is known [1] that a convex continuum is locally connected, and that local connectedness is a Whitney property [12, (14.9)]. Bing [1] and Moise [10] have shown independently that every locally connected continuum admits a convex metric. In view of these facts, we see that if \( X \) is a convex continuum, \( \mu^{-1}(t) \) admits a convex metric. However, as 3.2 shows, it may happen that \( \mu^{-1}(t) \) is not convex with respect to the Hausdorff metric.

References


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