EXAMPLES OF HEREDITARILY STRONGLY INFINITE-DIMENSIONAL COMPACTA

by

R. M. SCHORI AND JOHN J. WALSH
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Examples are given of strongly infinite dimensional compacta where each non-degenerate subcontinuum is also strongly infinite dimensional. These are by far the easiest of such examples in the literature and in addition a dimension theoretic phenomenon is identified which is used to verify this hereditary property.

1. Introduction

The first example of an infinite dimensional compactum containing no n-dimensional (n ≥ 1) closed subsets was given by D. W. Henderson [He] in 1967; shortly thereafter, R. H. Bing [Bi] gave a simplified version. In 1971, Zarelua [Z-1], in a relatively unknown article \(^2\) (in Russian), gives probably the simplest construction of this type of example. Later, in 1974, Zarelua [Z-2] constructed more complicated examples which had the property that each non-degenerate subcontinuum was strongly infinite dimensional. In 1977, the authors together with L. Rubin [R-S-W] developed an abstract dimension theoretic approach for constructing these types of examples; a significant feature of the latter approach was that the key concepts of essential families and continuum-wise separators were properly identified. The second author [Wa] used

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\(^2\)The authors only became aware of [Z-1] during the final draft of this paper.
this abstract approach to construct infinite dimensional compacta containing no n-dimensional \( n > 1 \) subsets (closed or not).

The examples presented in this paper have two important features: first, their construction is particularly simple and clearly illustrates the phenomena underlying all the previous constructions; and second, in spite of the simplicity of their construction, these examples have the property that every non-degenerate subcontinuum is strongly infinite dimensional. A phenomenon is isolated in §7 which shows that these examples are hereditarily strongly infinite dimensional and can be used to show that the "extra care" exercised in [Z-2] and [R-S-W] in order to insure this hereditary property is not necessary. The second example in this paper, see §6, uses the same construction as in [Z-1] where rather technical proofs are used to verify the weaker condition that the example contains no n-dimensional \( n > 1 \) closed subsets. This property follows rather automatically for us using the theory developed in [R-S-W].

2. Definitions and Basic Concepts

By a space we mean a separable metric space, by a compactum we mean a compact space, and by a continuum we mean a compact connected space. We follow Hurewicz and Wallman [H-W] for basic definitions and results in dimension theory. Specifically, by the dimension of a space \( X \), denoted \( \dim X \), we mean either the covering dimension or inductive dimension (since these are equivalent for separable metric spaces). A space which is not finite dimensional is said to be infinite.
We collect below the definitions and results needed in this paper; the reader is referred to [R-S-W] for a more thorough discussion.

2.1. **Definition.** Let A and B be disjoint closed subsets of a space X. A closed subset S of X is said to separate A and B in X if X-S is the union of two disjoint open sets, one containing A and the other containing B. A closed subset S of X is said to continuum-wise separate A and B in X provided every continuum in X from A to B meets S.

2.2. **Definition.** Let X be a space and \( \Gamma \) be an indexing set. A family \( \{ (A_k, B_k) : k \in \Gamma \} \) is essential in X if, for each \( k \in \Gamma \), \((A_k, B_k)\) is a pair of disjoint closed sets in \( X_k \) such that if \( S_k \) separates \( A_k \) and \( B_k \) in \( X \), then \( \cap \{ S_k : k \in \Gamma \} \neq \emptyset \).

2.3. **Theorem.** [H-W, p. 35 and p. 78]. For a space \( X \), \( \dim X \geq n \) if and only if there exists an essential family \( \{ (A_k, B_k) : k = 1, \ldots, n \} \) in \( X \).

2.4. **Remark.** Using the Hausdorff metric, the set of non-empty closed subsets of a compactum is a compactum. When we refer to a collection of closed subsets being dense, we mean dense with respect to the topology generated by this metric.

2.5. **Proposition.** [R-S-W; Proposition 3.4]. Let \( \{ (A_k, B_k) : k = 1, 2, \ldots, n \} \) be a collection of pairs of non-empty, disjoint closed subsets of a compactum \( X \). For each
k = 1,2,⋯,n, let $S_k$ be a non-empty dense set of separators of $A_k$ and $B_k$ and let $Y$ be a closed subset of $X$. If for each choice of separators $S_k \in S_k$, $k = 1,2,⋯,n$, we have that $(\cap \{ S_k : k = 1,2,⋯,n \}) \cap Y \neq \emptyset$, then $\{ (A_k \cap Y, B_k \cap Y) : k = 1,2,⋯,n \}$ is an essential family in $Y$ and, therefore, $\dim Y \geq n$.

2.6. Definition. A space $X$ is strongly infinite dimensional if there exists a denumerable essential family $\{ (A_k, B_k) : k = 1,2,⋯ \}$ for $X$. A space $X$ is hereditarily strongly infinite dimensional if each non-degenerate subcontinuum of $X$ is strongly infinite dimensional.

2.7. Theorem. [R-S-W; Proposition 5.5]. Let $X$ be a strongly infinite dimensional space with an essential family $\{ (A_k, B_k) : k = 1,2,⋯ \}$. For $k = 2,3,⋯$, let $S_k$ be a continuum-wise separator of $A_k$ and $B_k$ in $X$. If $Y = \cap \{ S_k : k = 2,3,⋯ \}$, then $Y$ contains a continuum meeting $A_1$ and $B_1$.

3. Outline of the Example

Let the Hilbert cube be denoted by $Q = \prod_I$ where $I_k = [0,1]$, let $\Pi_k : Q \rightarrow I_k$ denote the projection, and let $A_k = \Pi_k^{-1}(1)$ and $B_k = \Pi_k^{-1}(0)$. The family $\{ (A_k, B_k) : k = 1,2,⋯ \}$ is an essential family in $Q$ [H-W, p. 49].

For each $k = 1,2,⋯$, a space $Y_k = X_{3k-1} \cap X_{3k}$ will be constructed such that:

3.1. $X_j$ continuum-wise separates $A_j$ and $B_j$.

3.2. If $C$ is a closed subset of $Y_k$ and $\Pi_k(C) = I_k$, then $\dim C \geq 2$; if fact, $\{ (A_{3k-1} \cap C, B_{3k-1} \cap C), (A_{3k} \cap C, B_{3k} \cap C) \}$. 
is essential in C.

Thus, \( Y' = \cap \{ Y_k : k = 1,2,\ldots \} \) has the property guaranteed by Theorem 2.7 that \( Y' \) contains a continuum meeting \( A_1 \) and \( B_1 \) (also \( A_{3k+1} \) and \( B_{3k+1} \)) and if \( C \) is a closed subset of \( Y' \) such that for some \( k, \Pi_k(C) = I_k \), then \( \dim C \geq 2 \).

Also a space \( X_{3k+1} \) will be constructed such that 3.1 is satisfied as well as:

3.3. If \( C \) is a non-degenerate subcontinuum of \( Y'' = \cap \{ X_{3k+1} : k = 1,2,\ldots \} \), then there is an integer \( k \) such that \( \Pi_k(C) = I_k \).

The space \( Y = Y' \cap Y'' = \cap \{ X_k : k = 2,3,\ldots \} \) will be an example of a hereditarily strongly infinite dimensional space. We will now argue using conditions 3.1-3.3 that it is an infinite dimensional compactum that contains no \( n \)-dimensional \( (n \geq 1) \) closed subsets. Theorem 2.7 guarantees that \( Y \) contains a continuum meeting \( A_1 \) and \( B_1 \) and hence \( \dim Y \geq 1 \), and 3.2 and 3.3 guarantee that \( X \) contains no 1-dimensional subcontinua. Then the compactness insures that \( X \) contains no 1-dimensional closed subsets since compact totally disconnected sets are 0-dimensional. This is sufficient since, from the inductive definition of dimension, it is clear that each closed \( n \)-dimensional \( (n \geq 1) \) set contains \( k \)-dimensional closed subsets for each \( 0 \leq k < n \) and in particular for \( k = 1 \). Thus, \( Y \) is infinite dimensional and contains no \( n \)-dimensional \( (n \geq 1) \) closed subsets. In section 6 we prove that this example is hereditarily strongly infinite dimensional.

4. Constructing \( Y_k \)

Let \( \{ W_i : i = 1,2,\ldots \} \) be the null sequence of open
intervals in $I_k$ indicated in Figure 1. Let $\{S_{3k-1}^i: i = 1,2,\ldots\}$ and $\{S_{3k}^i: i = 1,2,\ldots\}$ be a countable dense sets of separators of $A_{3k-1}$ and $B_{3k-1}$ and $A_{3k}$ and $B_{3k}$, respectively. Let $a: N \to N \times N$ be a bijection where $N$ denotes the natural numbers and let $a_1$ and $a_2$ be $a$ composed with projection onto the first and second factor, respectively.

Let $X_{3k-1} = \pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\ldots\}) \cup (\cup(S_{3k-1}^i \cap \pi_k^{-1}(W_i): i = 1,2,\ldots))$ and let $X_{3k} = \pi_k^{-1}(I_k - \cup\{W_i: i = 1,2,\ldots\}) \cup (\cup(S_{3k}^i \cap \pi_k^{-1}(W_i): i = 1,2,\ldots))$; see Figure 2.

where $k = 1$. It is easily seen that $X_{3k-1}$ and $X_{3k}$ continuum-wise separate $A_{3k-1}$ and $B_{3k-1}$ and $A_{3k}$ and $B_{3k}$, respectively.

In addition, if $C \subseteq X_{3k-1} \cap X_{3k}$ with $\pi_k(C) = I_k$ and $(i,j) \in N \times N$, then $C \cap \pi_k^{-1}(W_i) \subseteq S_{3k-1}^i \cap S_{3k}^j$; therefore, Proposition 2.5 guarantees that if $C$ is a closed subset of $X_{3k-1} \cap X_{3k}$ with $\pi_k(C) = I_k$, then $\dim C \geq 2$.

The nature of $X_{3k+1}$ is different than that of $X_{3k-1}$ and $X_{3k}$; the role of $X_{3k+1}$ is to insure that condition 3.3 will hold. Let $X_{3k+1} = \pi_k^{-1}(R_{3k+1})$ where $\pi_k,R_{3k+1}$ is the projection onto $I_k \times I_{3k+1}$ and $R_{3k+1} \subseteq I_k \times I_{3k+1}$ is the "roof-top" in Figure 3.
5. Verifying Condition 3.3

If $J$ is a subinterval of $[0,1]$, let $\ell(J)$ denote the length of $J$. Let $C \subseteq Y$ be a non-degenerate subcontinuum, let $i_1$ be such that $\Pi_{i_1}(C)$ is also non-degenerate, and let $\ell(\Pi_{i_1}(C)) = \varepsilon > 0$. Note that since the slopes of the straight line segments of $R_{3i+1}$ are $\pm 2$, and $C \subseteq X_{3i+1}$, then $\frac{1}{2} \notin \Pi_{i_1}(C)$.
implies that \( \ell(\Pi_{3i+1}(C)) = 2 \varepsilon \). Inductively, let \( i_n = 3i_{n-1} + 1 \), let \( J_n = \Pi_{i_n}(C) \) and observe that if \( \frac{1}{2} \notin J_{n-1}' \), then \( \ell(J_n) = n\varepsilon \). Since each \( J_n \) has length 1, it follows that there exists an \( N \) such that \( \frac{1}{2} \in J_N \). Thus, by observing the corresponding properties of \( R_{3i+1} \), it follows that \( 1 \notin J_{N+1} \) and that \( 0 \in J_{N+2} = [0, b] \) for some \( 0 < b < 1 \). Following the above argument we see that if \( \frac{1}{2} \leq b < 1 \), then \( J_{N+3} = [0, 1] \) and if \( 0 < b < \frac{1}{2} \), then \( J_{N+3} = [0, 2b] \) and hence for some \( j > 3 \), \( J_{N+j} = [0, 1] \) which says that for some \( k \), \( \Pi_k(C) = I_k \).

6. A Generalization

Let \( X \) be a strongly infinite dimensional compactum with essential family \( \{(A_k, B_k) : k = 1, 2, \ldots \} \); let \( \{\Pi_k : k = 1, 2, \ldots \} \) be a countable dense subset of the space of all mappings from \( X \) to \( I = [0, 1] \); for each \( k \), let \( \{S^k_i : i = 1, 2, \ldots \} \) be a countable dense set of separators of \( A_k \) and \( B_k \), and let \( \{W_i : i = 1, 2, \ldots \} \) be the null sequence of open intervals in \([\frac{1}{3}, \frac{2}{3}]\) indicated in Figure 4.

\[
\begin{align*}
0 & \quad \frac{1}{2} & \quad \frac{2}{3} & \quad 1 \\
\cdots W_3 & W_2 & W_1 \\
& \quad \text{Fig. 4}
\end{align*}
\]

Let \( a, a_1, a_2 \) be as before and, for each \( k \), let \( Y_k = X_{2k} \cap X_{2k+1} \) where

\[
x_{2k} = \Pi_{-1}^k(I_k - \cup\{W_i : i = 1, 2, \ldots \}) \cup \left( \cup\{S^2_{a_1(i)} \cap \Pi_k^{-1}(W_i) : i = 1, 2, \ldots \} \right)
\]

and
\[ X_{2k+1} = \Pi_k^{-1}(I_k - \bigcup \{W_i : i = 1, 2, \ldots \}) \cup \\
(\bigcup S_{2k+1}^{2k+1} \cap \Pi_k^{-1}(W_i) : i = 1, 2, \ldots ). \]

It is easily seen that condition 3.1 is true and the earlier argument shows that:

6.1. If \( C \) is a closed subset of \( Y_k \) and \( \Pi_k(C) \supseteq [\frac{1}{2}, \frac{3}{2}] \),
then \( \dim C \geq 2 \), if fact, \( \{ (A_{2k} \cap C, B_{2k} \cap C), (A_{2k+1} \cap C, B_{2k+1} \cap C) \} \) is essential in \( C \).

Letting \( Y = \cap \{ Y_k : k = 1, 2, \ldots \} = \cap \{ X_k : k = 2, 3, \ldots \} \),
Theorem 2.7 guarantees that \( Y \) contains a continuum meaning \( A_1 \) and \( B_1 \). Since the \( \Pi_k \)'s are a dense set of mappings the following holds:

6.2. If \( C \subseteq Y \) is a non-degenerate subcontinuum of \( Y \),
then for some \( k, \Pi_k(C) \supseteq [\frac{1}{2}, \frac{3}{2}] \).

Thus our previous argument shows that we have constructed in an arbitrary strongly infinite dimensional space \( X \) a subcompactum \( Y \) that is infinite dimensional and contains no \( n \)-dimensional \( (n > 1) \) closed subsets. We will show in the next section that in fact \( Y \) is hereditarily strongly infinite dimensional.

7. Strong Infinite Dimensionality of Subcontinua

One reason for the additional complexity in the construction in [Z-2] and [R-S-W] was to be able to conclude that the examples had the additional property that each non-degenerate subcontinuum was strongly infinite dimensional. Although we made no effort to construct examples with this hereditary property, the following propositions isolate a phenomenon which forces them to have this property.
Proposition 7.1 gives conditions on a continuum that imply it is strongly infinite dimensional. Observe that conditions 3.2 and 3.3 (resp., 6.1 and 6.2) imply that each non-degenerate subcontinuum of the example constructed in sections 3 and 4 (resp., section 6) satisfies the hypothesis of Proposition 7.1 and thus these examples are hereditarily strongly infinite dimensional. An alternative argument for the example constructed in section 6 can be given using Proposition 7.2.

7.1. Proposition. Let \( \{(A_k, B_k) : k = 1, 2, \ldots \} \) be a family of pairs of disjoint closed subsets of a continuum \( X \). Suppose that, for each \( k \), there are positive integers \( i \) and \( j \) such that, for each continuum \( C \subseteq X \) meeting \( A_k \) and \( B_k \), the pair \( \{(A_i \cap C, B_i \cap C), (A_j \cap C, B_j \cap C)\} \) is essential in \( C \).

If, for some \( n \), \( A_n \neq \emptyset \) and \( B_n \neq \emptyset \), then \( X \) is strongly infinite dimensional. Alternately, if for some \( i \) and \( j \), \( \{(A_i \cap X, B_i \cap X), (A_j \cap X, B_j \cap X)\} \) is essential in \( X \), then \( X \) is strongly infinite dimensional.

Proof. Let \( i_1 \) and \( j_1 \) be such that \( \{(A_{i_1}, B_{i_1}), (A_{j_1}, B_{j_1})\} \) is essential in \( X \). Let \( i_2 \) and \( j_2 \) be such that for each continuum \( C \) meeting \( A_{i_1} \) and \( B_{j_1} \), \( \{(A_{i_2} \cap C, B_{i_2} \cap C), (A_{j_2} \cap C, B_{j_2} \cap C)\} \) is essential in \( C \). Recursively, for \( n \geq 3 \), let \( i_n \) and \( j_n \) be such that for each continuum \( C \) meeting \( A_{j_{n-1}} \) and \( B_{j_{n-1}} \), \( \{(A_{i_n} \cap C, B_{i_n} \cap C), (A_{j_n} \cap C, B_{j_n} \cap C)\} \) is essential in \( C \). We now show that the family \( \{(A_{i_n}, B_{i_n}) : n = 1, 2, \ldots \} \) is essential in \( X \). For \( n = 1, 2, \ldots \), let \( S_n \) separate \( A_{i_n} \) and \( B_{i_n} \).
Since \( \{(A_1, B_1), (A_2, B_2)\} \) is essential in \( X \), \( S_1 \) contains a continuum from \( A_1 \) to \( B_1 \). Since \( \{(A_2, B_2), (A_1, B_1)\} \) is essential in this continuum, \( S_1 \cap S_2 \) contains a continuum from \( A_2 \) to \( B_2 \). Since \( \{(A_1, B_1), (A_2, B_2)\} \) is essential in this continuum, \( S_1 \cap S_2 \cap S_3 \) contains a continuum from \( A_3 \) to \( B_3 \). Continuing this argument, for each \( n \geq 1 \), \( S_1 \cap \cdots \cap S_n \) contains a continuum from \( A_n \) to \( B_n \) and, therefore, \( \cap\{S_n: n = 1, 2, \cdots\} \neq \emptyset \).

7.2. Proposition. Let \( X \) be a compactum with \( \dim X \geq 1 \). Suppose that, for each pair of disjoint closed sets \( H \) and \( K \), there is a family \( \{(A, B), (D, E)\} \) of pairs of disjoint closed sets such that \( \{(A \cap C, B \cap C), (D \cap C, E \cap C)\} \) is essential in each continuum \( C \) from \( H \) to \( K \). Then each non-degenerate subcontinuum of \( X \) is strongly infinite dimensional.

**Proof.** Since the hypotheses are satisfied by non-degenerate subcontinua of \( X \), it suffices to assume that \( X \) is a continuum and to show that \( X \) is strongly infinite dimensional. Let \( \{(A_1, B_1), (D_1, E_1)\} \) be an essential family in \( X \). Recursively, for \( n \geq 2 \), let \( \{(A_n, B_n), (D_n, E_n)\} \) be such that \( \{(A_n \cap C, B_n \cap C), (D_n \cap C, E_n \cap C)\} \) is essential in each continuum \( C \) from \( D_{n-1} \) to \( E_{n-1} \). The argument used in the proof of Proposition 7.1 shows that \( \{(A_n, B_n): n = 1, 2, \cdots\} \) is essential in \( X \).

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