SOME REMARKS ON M-EMBEDDING

by

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Section 1

There are four main results in this paper: (1) a necessary condition for the product of a space with any metric space to be normal, (2) a characterization of compact $T_2$ spaces, (3) a complete analogue of the Morita-Hoshina Homotopy Extension Theorem (3.7 [13]) for ANR spaces, and (4) a characterization of spaces for which every metric space is an AE. Each of these results involves the notion of M-embedding, which was introduced in [17]. (See also [8], [15])

In what follows, $\gamma$ will denote an infinite cardinal number, $\mathbb{R}$ will denote the reals, $\mathbb{P}$ the irrationals, and $I$ the unit interval; all functions and pseudometrics will be assumed continuous. No separation axioms will be assumed unless stated.

We say a subspace $S$ of a topological space $X$ is $M^\gamma$-embedded ($P^\gamma$-embedded) in $X$ if every function from $S$ to a $\gamma$-separable (complete) metrizable AE extends to $X$. By an AE or ANR we mean an AE or ANR for metric spaces. By dropping the separability condition, we obtain definitions of $P$- and $M$-embedding. $P$-embedding has been extensively studied, for example, see [1, 2, 13, 14]. For definitions of $C^*$- and $C$-embedding see [6].

There are certain results we will frequently use, and we list them here.

(a) $S$ is $P^\gamma$-embedded ($M^\gamma$-embedded) in $X$ iff every function
from S to a \(\gamma\)-separable Banach space (normed linear space) extends to X (p. 227 [1], Th. 1 [17]).

(8) X is \(\gamma\)-collectionwise normal iff every closed subset is \(P^\gamma\)-embedded in X (p. 189 [1]).

(\(\delta\)) S is \(P^{NO}\)-embedded in X iff S is C-embedded in X (p. 200 [1]).

(\(\eta\)) S is \(M^\gamma\)-embedded in X iff S is \(P^\gamma\)-embedded in X and given a \(\gamma\)-separable pseudometric \(d\) on X, there exists a zero set \(Z\) of X such that \(S \subseteq Z \subseteq \{x \in X : d(x,x_0) = 0 \text{ for some } x_0 \in S\}\) (Th. 1 [17]).

(\(\theta\)) S is \(M^\gamma\)-embedded in X iff S is \(P^\gamma\)-embedded in X and given a function \(f\) from X to a \(\gamma\)-separable metric space, there exists a zero set \(Z\) of X such that \(S \subseteq Z \subseteq f^{-1}f(S)\) (Th. 1 [17]).

(\(\kappa\)) S is \(P^\gamma\)-embedded in X iff \(S \times Y\) is \(P^\gamma\)-embedded in \(X \times Y\) for every compact \(T_2\) space \(Y\) with \(w(Y) \leq \gamma\) (p. 234 [1] (X need not be \(T_{3\frac{1}{2}}\))).

(\(\lambda\)) S is \(P^\gamma\)-embedded in X iff \(S \times Y\) is \(C^*\)-embedded in \(X \times Y\) for every compact \(T_2\) space \(Y\) with \(w(Y) \leq \gamma\) (p. 234 [1]--for a sharpened version see [14]).

Removing the cardinality restrictions on each of these (except (\(\delta\))) produces characterizations of \(P\)- and \(M\)-embedding and of collectionwise normality.

**Section 2**

Since \(M^{NO}\)-embedding (\(P^{NO}\)-embedding) is equivalent to the extendability of every function into a separable (complete) metrizable AE and since \(P^{NO}\)-embedding is equivalent to C-embedding (fact (\(\delta\)) of Section 1), one might wonder whether S
is $M^{H_0}$-embedded in $X$ iff (*): every function from $S$ into an AE embedded in $R$ extends to $X$. Note that a subset of $R$ is an AE iff it is an interval. 2.1 will show that the above conjecture is false as (*) is equivalent to C-embedding. Example 2.4 of [8] (identical with the example on p. 224 of [17]) shows that C-embedding is strictly weaker than $M^{H_0}$-embedding. (2.1 was first shown by R. Arens for closed subsets of normal spaces [2].)

2.1 Proposition. If $S$ is C-embedded in $X$, every function from $S$ to an interval $K$ of $R$ extends to $X$ with values in $K$.

Proof. There is an extension $g$ of $f$ with $g(X) \subseteq K$. Assuming that $K$ is not closed, $K - K$ consists of 1 or 2 points and hence is a zero set of $R$. Hence $g^{-1}(K - K)$ is a zero set of $X$ disjoint from $S$. Hence there exists $h: X \to [0,1]$ such that $h(S) \equiv 1$ and $h(g^{-1}(K - K)) \equiv 0$ (p. 19 [6]). Fix $r \in K$ and define $f^* = hg + (1 - h)r$.

This same idea will work if $S$ is $P^\gamma$-embedded in $X$ and $f$ is a function from $S$ to a convex subset $K$ of a $\gamma$-separable Banach space $B$ such that $K - K$ is a zero set in $B$. (See 4.1 [2])

Fact (6) of Section 1 with $\gamma = H_0$ tells us that $S$ is $M^{H_0}$-embedded in $X$ iff it is $P^{H_0}$-embedded and given a function $f$ from $X$ to a separable metric space, there exists a zero set $Z$ of $X$ such that $S \subseteq Z \subseteq f^{-1}f(S)$. One might ask whether $M^{H_0}$-embedding is equivalent to C-embedding plus (**): Given $f: X \to R$, there exists a zero set $Z$ of $X$ such that $S \subseteq Z \subseteq f^{-1}f(S)$. The answer is again no.

To see this, let $X$ be the unit disc in the plane (as a
set). And \( S = \{(x,y) : x^2 + y^2 < 1, \text{ or } x^2 + y^2 = 1 \text{ and } x \text{ is rational}\}. \) Let \( X \) have the topology that makes the points of \( X - S \) discrete. Hence open sets of \( X \) are of the form \( U \cup V \), where \( U \) is an open neighborhood in the ordinary metric topology and \( V \) is a subset of \( X - S \). Any space formed in this way is hereditarily paracompact (see [10]). Hence \( S \) is a closed \( C \)-embedded subset of \( X \). Since \( S \) is an AR that is not an absolute \( C_\delta \) (see p. 382 [7]), we can show that \( S \) is not a zero set of \( X \). Since \( X \) is submetrizable (i.e. its topology contains a metric topology), it is clear from \((\eta)\) in Section 1 that \( S \) is not \( M^{\infty}_0 \)-embedded in \( X \). (To see this, let \( d \) be the metric topology on \( X \).) However, let \( f : X \rightarrow R \) and observe that since \( S \) is connected, \( f(S) \) is an interval and hence is a \( G_\delta \). Therefore \( f^{-1}f(S) \) is a \( G_\delta \) set of \( X \) containing \( S \); since \( X \) is normal, there exists a zero set \( Z \) such that \( S \subseteq Z \subseteq f^{-1}f(S) \).

Section 3

There is considerable interest in spaces whose product with every metric space is normal. A characterization of this class was given by Morita [11, 12]. A theorem due to Morita, Rudin, and Starbird states that if \( Y \) is metric and \( X \) normal and countably paracompact, then \( X \times Y \) is normal iff \( X \times Y \) is countably paracompact [16].

This section will produce a necessary condition for the product of a normal space \( X \) with every \( \gamma \)-separable metric space to be normal. If \( S \) is a subspace of \( X \), we say that \((X,S)\) has the \( \gamma \)-Zero-Set Interpolation Property (\( \gamma \)-ZIP) if whenever \( d \) is a \( \gamma \)-separable pseudometric on \( X \), there exists
a zero set $Z$ of $X$ such that:

$$S \subset Z \subset \{x \in X: d(x, x_o) = 0 \text{ for some } x_o \in S\}.$$  

By (n) in Section 1, we see that $S$ is $M^\gamma$-embedded in $X$ iff $S$ is $P^\gamma$-embedded and $(X, S)$ has the $\gamma$-ZIP. Hence the $\gamma$-ZIP is what needs to be added to $P^\gamma$-embedding to produce $M^\gamma$-embedding. By dropping the separability condition on $d$, we obtain a definition of the Zero-Set Interpolation Property (ZIP), and observe that $S$ is $M$-embedded in $X$ iff $S$ is $P$-embedded in $X$ and $(X, S)$ has the ZIP. The following proposition is a slight generalization of an example communicated to the author by E. Michael (the example is written up in Section 3 of [18]).

3.1 Proposition. Let $S$ be a closed subset of a normal space $X$ such that $S \times Y$ is $C$-embedded in $X \times Y$ for every $\gamma$-separable metric space $Y$. Then $(X, S)$ has the $\gamma$-ZIP.

Proof. Let $d$ be a $\gamma$-separable pseudometric on $X$ and let $A = \{x \in X: d(x, x_o) = 0 \text{ for some } x_o \in S\}$. Let $(Y, d)$ be the $\gamma$-separable metric space associated with the pseudometric space $(X - A, d)$. For notational ease we will identify points of $Y$ with those of $X - A$. Define $f: S \times Y \to R$ by $f(x, y) = 1/d(x, y)$. The map $f$ is well-defined and continuous hence extends to $g: X \times Y \to R$.

Let $H_n = \{x \in X - A: d(x, y) < 1/n \Rightarrow g(x, y) < n\}$. We claim $X - A = \bigcup_n H_n$. Let $x_o \in X - A$ and choose $m$ such that $g(x_o, x_o) < m$. Since $g$ is continuous there exists an open set $U$ of $X$ containing $x_o$ and an $\varepsilon > 0$ such that if $x \in U$ and $d(x, x_o) < \varepsilon$, then $g(x, y) < m$. Choose $n$ such that $n > m$ and $1/n < \varepsilon$. Then $x_o \in H_n$.

Hence we have $H = \bigcap_n (X - H_n) \subset A$. We claim that $S \subset H$.  

This will finish the proof, for since $S$ is closed, $X$ is normal, and $H$ is a $G_\delta$, we will be able to find a zero set $Z$ such that $S \subseteq Z \subseteq A$. To show that $S \subseteq H$, argue by contradiction. Assume there exists $x_\circ \in S \cap \overline{H}_n$ for some $n$. Choose $y_\circ \in X - A$ such that $d(x_\circ, y_\circ) < 1/2n$. (We can do this since the topology generated by $d$ is contained in the topology on $X$ and $x_\circ \in \overline{H}_n$.) Then $g(x_\circ, y_\circ) > 2n$. Since $g$ is continuous, there exists an open $U$ containing $x_\circ$ and $\varepsilon > 0$ such that $x \in U$ and $y \in Y$ with $d(y, y_\circ) < \varepsilon$ implies $g(x, y) > n$.

Choose $x \in U \cap H_n$ such that $d(x, x_\circ) < 1/2n$. Then $d(x, y_\circ) \leq d(x, x_\circ) + d(x_\circ, y_\circ) < 1/n$, hence $g(x, y_\circ) < n$ (since $x \in H_n$). However, $x \in U$ and hence $g(x, y_\circ) > n$, which is the desired contradiction.

There are a number of corollaries of this result. For example:

3.2 Corollary. If $X \times Y$ is normal for every metric $Y$, then every closed subset of $X$ has the ZIP with respect to $X$.

3.3 Corollary. If $X \times Y$ is normal for every separable metric $Y$, then every closed subset of $X$ is $\mathfrak{M}_\mathfrak{O}$-embedded in $X$.

Proof. Use (8) and ($\eta$) of Section 1.

3.4 Corollary. Let $X$ be a collectionwise normal space whose product with every metric space is normal. Then every closed subset of $X$ is $\mathfrak{M}$-embedded in $X$.

Proof. Use (8) and ($\eta$) of Section 1.

3.5 Corollary (Michael). The following are equivalent for a submetrizable space $X$:
(a) $X$ is perfectly normal.

(b) $X \times Y$ is perfectly normal for every metric $Y$.

(c) $X \times Y$ is normal for every metric $Y$.

Proof. (b) $\Rightarrow$ (c) is clear and (a) $\Rightarrow$ (b) is known [9].

Hence we need only show (c) $\Rightarrow$ (a). Assume (c) but suppose (a) fails. This implies that $X$ is normal and submetrizable, but not perfectly normal. From the definition of the ZIP, it is clear that a subset of a submetrizable space $X$ has the ZIP with respect to $X$ iff it is a zero set. (One may see this by letting $d$ be the metric whose topology is contained in that of $X$.) Hence $X$ contains a closed subset $S$ such that $(X, S)$ fails to have the ZIP, so by 3.1 there exists a metric space $Y$ such that $X \times Y$ is not normal, giving a contradiction.

In fact, it is clear from the above that if $X$ contains a $\gamma$-separable metric topology and fails to be perfectly normal, then there exists a $\gamma$-separable metric $Y$ such that $X \times Y$ is not normal. More specifically, if $m$ is a continuous metric on $X$ and $S$ is a closed non-$G_\delta$ subset of $X$, then $S \times Y$ fails to be $C$-embedded in $X \times Y$, where $Y$ is the metric space $(X - S, m)$. This shows immediately that $X \times P$ fails to be normal, where $X$ is the Michael line and $P$ the irrationals with their usual topology. A different proof was originally given in [10].

It is an open question whether the converse of 3.1 is true. 4.5 of Section 4 will shed some light on this question.

Section 4

Morita and Hoshina (Theorem 3.7 [13]) proved the
following generalization of the Homotopy Extension Theorem:

4.1 Theorem. For a subspace $S$ of a topological space $X$ the following are equivalent:

1) $S$ is $P_Y$-embedded in $X$.

2) $(S \times Y) \cup (X \times B)$ is $P_Y$-embedded in $X \times Y$ for every compact $T_2$ space $Y$ with $w(Y) \leq \gamma$ and its closed subset $B$.

3) $(S \times I) \cup (X \times \{0\})$ is $P_Y$-embedded in $X \times I$.

4) $(X,S)$ has the HEP with respect to every complete ANR space of weight $\leq \gamma$.

The analogue of 4.1 for $M_Y$-embedding is the following:

4.2 Theorem. The following are equivalent:

1) $S$ is $M_Y$-embedded in $X$.

2) $(S \times Y) \cup (X \times B)$ is $M_Y$-embedded in $X \times Y$ for every compact $T_2$ space $Y$ with $w(Y) \leq \gamma$ and its closed subset $B$.

3) $(S \times I) \cup (X \times \{0\})$ is $M_Y$-embedded in $X \times I$.

4) $(X,S)$ has the HEP with respect to every ANR space of weight $\leq \gamma$.

Proof. The equivalence of (1), (3), and (4) is Theorem 2 of [17]. To complete the proof it remains to show (1) $\Rightarrow$ (2). We state and prove the next theorem, then use it to show (1) $\Rightarrow$ (2).

4.3 Theorem (L. Sennott, R. Levy, M. D. Rice). The following are equivalent for a $T_2$ space $Y$:

1) The space $Y$ is compact.
(2) If $Y$ is embedded in a $T_{3\frac{1}{2}}$ space $Z$ and $X$ is any space, then $X \times Y$ is $M$-embedded in $X \times Z$.

(3) If $Y$ is embedded in a $T_{3\frac{1}{2}}$ space $Z$ and $X$ is any space, then $X \times Y$ is $C^*$-embedded in $X \times Z$.

Proof. To show (1) \Rightarrow (2) let $Y$ be a compact space embedded in a $T_{3\frac{1}{2}}$ space $Z$, let $X$ be any space, and let $f: X \times Y \to L$ be a continuous function into a normed linear space $L$. By (a) of Section 1, it is sufficient to extend $f$ to $X \times Z$. Define $g: X \to C^*(Y,L)$ by $g(x)(y) = f(x,y)$. A standard argument shows that $g$ is continuous when $C^*(Y,L)$ has the sup norm topology. We then define $h: g(X) \times Y \to L$ by $h(g(x),y) = f(x,y)$ and observe that $g(X)$ is a metric space and $h$ is continuous. Now $g(X) \times \beta Z$ is the product of a metric space and a compact space and hence is a paracompact M-space. This implies that the closed subset $g(X) \times Y$ is $M$-embedded (Proposition 2 of [17]). Hence we can lift $h$ to $h^*: g(X) \times \beta Z \to L$. Defining $f^*: X \times Z \to L$ by $f^*(x,z) = h^*(g(x),z)$, one checks that this defines a continuous extension of $f$.

Note: This proof uses an idea contained in the proof of Theorem 2 of [19] and in fact M. Starbird's Theorem 3 [19] is our (1) \Rightarrow (3) with $C^*$-embedding replaced by $C$-embedding.

Clearly (2) \Rightarrow (3). Now assume (3) holds but $Y$ is not compact. By Problem 6J of [6], the space $Y$ is absolutely $C^*$-embedded and hence is almost compact. Let $\beta Y - Y = \{\omega\}$, and let $\{U_\alpha: \alpha \in D\}$ be a base of open neighborhoods of $\omega$ in $\beta Y$. We will define a space $X$ such that $X \times Y$ is not $C^*$-embedded in $X \times \beta Y$. Define an ordering on $D$: $\alpha \leq \beta$ iff
U_\beta \subseteq U_\alpha. Then D becomes a directed set. Let X = D \cup \{q\}, where q \notin D, points of D are isolated and basic open neighborhoods of q are of the form \{q\} \cup \{\alpha: \alpha \geq \alpha_0\}. Denote this set by [\alpha_0, q].

For each \alpha, choose a function f_{\alpha} on \beta Y such that f_{\alpha}(\beta Y - U_\alpha) is identically 1 and f_{\alpha}(\infty) = 0. Define f: X \times Y \to [0,1] by f(\alpha,y) = f_{\alpha}(y) and f(q,y) = 1. Clearly f is continuous at points of the form (\alpha,y). Fix (q,y_0) and choose \alpha_0 such that y_0 \notin \overline{U}_{\alpha_0}. If (x,y) \in [\alpha_0, q] \times (Y - \overline{U}_{\alpha_0}), then f(x,y) = 1.

If there were an extension of f to all of X \times \beta Y, the extension would be 1 at all points of the form (q,y) with y \in Y and 0 at all points (\alpha,\infty), which implies that the extension is not continuous at (q,\infty).

Note: This proof is a generalization of an example given by Comfort and Negrepontis (Example 4.6 of [4]).

To complete the proof of 4.2, let S be M^Y-embedded in X, and Y and B be as in (2). By 4.3 (2) it is clear that X \times B is M^Y-embedded in X \times Y. By proposition 5 of [17], we have that S \times Y is M^Y-embedded in X \times Y. By Proposition 6 of [17], to show (2) it is sufficient to show that (S \times Y) \cup (X \times B) is P^Y-embedded in X \times Y. But this is true from (1) \Rightarrow (2) of 4.1.

We now use 4.3 to obtain a generalization of (K) in Section 1, which will throw further light on the results in Section 3.

4.4 Proposition. The following are equivalent:
(1) $S$ is $P_Y$-embedded in $X$.

(2) $S \times Y$ is $P_Y$-embedded in $X \times Y$ for every locally compact, paracompact $T_2$ space $Y$ with $w(Y) \leq \gamma$.

(3) $S \times Y$ is $C^*$-embedded in $X \times Y$ for every locally compact, paracompact $T_2$ space $Y$ with $w(Y) \leq \gamma$.

Proof. $(2) \Rightarrow (3)$ is clear and $(3) \Rightarrow (1)$ is clear from $(\lambda)$ of Section 1. It remains to show $(1) \Rightarrow (2)$. Let $S$ be $P_Y$-embedded in $X$ and $Y$ as in $(2)$. For each $y \in Y$, let $U_y$ denote an open neighborhood of $y$ whose closure is compact. Let $\{f_\alpha : \alpha \in A\}$ be a locally finite partition of unity subordinate to the cover $\{U_y : y \in Y\}$, and let $K_\alpha$ denote the compact set $\text{cl}(Y - Z(f_\alpha))$.

Let $g : S \times Y \to B$ be a function into a $\gamma$-separable Banach space $B$. By $(\alpha)$ of Section 1, it is sufficient to extend $g$ to $X \times Y$. For each $\alpha$, the function $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha : X \times K_\alpha \to B$ by $(\kappa)$ of Section 1. By 4.3 (2), $h_\alpha$ extends to $k_\alpha : X \times Y \to B$. Then $g^*(x,y) = \sum_{\alpha} f_\alpha(y)k_\alpha(x,y)$ is the desired extension of $g$.

4.5 Corollary. If $S$ is $C$-embedded in $X$, then $S \times Y$ is $C$-embedded in $X \times Y$ for any locally compact metric space $Y$.

Proof. Let the compact sets $K_\alpha$ be constructed as in the proof of $(1) \Rightarrow (2)$ of 4.4. If $Y$ is metric, then $K_\alpha$ is compact metric. Let $g : S \times Y \to R$. By $(\delta)$ and $(\kappa)$ of Section 1, $g_\alpha = g|S \times K_\alpha$ has an extension to $h_\alpha : X \times K_\alpha \to R$. The proof proceeds as in 4.4.

Comparing 3.1 and 4.5, we see that if $S$ is a closed subset of a normal space $X$ such that $(X,S)$ fails to have ZIP, then there exists a non-locally compact metric space $Y$ such
that $S \times Y$ is not C-embedded in $X \times Y$.

4.6 Corollary. If $S$ is $M^\gamma$-embedded in $X$, then $S \times Y$ is $M^\gamma$-embedded in $X \times Y$ for any locally compact paracompact $T_2$ space $Y$ with $w(Y) \leq \gamma$.

Proof. The proof of $(1) \Rightarrow (2)$ of 4.4 goes through with $B$ replaced by a $\gamma$-separable normed linear space. (To lift $q_\alpha$ use 4.2 $(1) \Rightarrow (2)$.)

Section 5

As a final application of $M$-embedding, we generalize two results of E. Chang [3]. (Also see results of Ellis [5].) Although the results deal with ultranormal spaces, they are equivalent to the following:

5.1 Proposition (Chang, p. 38, 40 [3]). Let $X$ be nonempty. The following are equivalent.

(1) $X$ is a $0$-dim collectionwise normal (normal) space.

(2) Every complete (separable) metric space is an $AE$ for $X$.

5.2 Proposition (Chang, p. 43 [3]). Let $S$ be a closed $G_\delta$ subset of a $0$-dim collectionwise normal (normal) space $X$, $Y$ a (separable) metric space and $f: S \rightarrow Y$. Then $f$ extends to $X$.

5.3 Proposition. Let $X$ be nonempty. The following are equivalent.

(1) Every (separable) metric space is an $AE$ for $X$.

(2) $X$ is a $0$-dim space in which every closed subset is $M^{HO}$-embedded.
Proof. We prove the unbracketed equivalence. \((1) \Rightarrow (2)\) is clear from 5.1 and the definition of \(M\)-embedding. To show \((2) \Rightarrow (1)\), let \(Y\) be a metric space, \(S\) a closed subset of \(X\), and \(f: S \rightarrow Y\). Let \(\bar{Y}\) denote the completion of \(Y\) with injection map \(i\). Since \(X\) is a 0-dim collectionwise normal space, the map \(i \circ f: S \rightarrow \bar{Y}\) has an extension \(\bar{f}\) to \(X\), by 5.1. By (\(\Theta\)) of Section 1, there exists a zero set \(Z\) of \(X\) such that \(S \subseteq Z \subseteq f^{-1}(\bar{f}(S))\). Hence \(\bar{f}|_Z\) maps \(Z\) into \(Y\), so by 5.2 it can be lifted to \(f^*: X \rightarrow Y\), completing the proof.

In [15], Morita remarks that the following generalizations of known results can be proved: If \(\dim X/S \leq n+1\), then \(S\) is \(M^{\gamma}\)-embedded (\(P^{\gamma}\)-embedded) in \(X\) iff any map from \(S\) into a metric (complete metric) space of weight \(\leq \gamma\) which is \(LC^n\) and \(C^n\) can be extended to \(X\). If \(\dim X/S \leq n\), then \(S\) is \(M^{\gamma}\)-embedded (\(P^{\gamma}\)-embedded) in \(X\) iff \((X,S)\) has the homotopy extension property with respect to every metric (complete metric) space of weight \(\leq \gamma\) which is \(LC^n\).

References


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