TOPOLOGY PROCEEDINGS

Volume 4, 1979

Pages 13–17

http://topology.auburn.edu/tp/

CECH-STONE REMAINDERS OF LOCALLY COMPACT NONPSEUDOCOMPACT SPACES

by

Robert L. Blair

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

ČECH-STONE REMAINDERS OF LOCALLY COMPACT NONPSEUDOCOMPACT SPACES

Robert L. Blair

All spaces considered are Tychonoff, X* denotes the Čech-Stone remainder $\beta X - X$, and N, Q, and R denote the spaces of natural numbers, rationals, and reals, respectively. A subset S of a space X is *z-embedded* in X if every zero-set of S is the restriction to S of some zero-set of X, and X is an *Oz-space* if every open subset of X is z-embedded in X. (For example, extremally disconnected spaces and perfectly normal spaces are Oz. For studies involving Oz-spaces, see [B], [vD_4], [La_1], [La_2], [T_1], and [T_2].)

In [B, 5.13] we proved the following (which generalizes the well-known fact that N* is not extremally disconnected [GJ, 6R.1]):

Theorem 1. *If* $|X|$ *is not Ulam-measurable and if* X *is locally compact and realcompact but not compact, then* X* *is not* Oz.

The proof in [B] of Theorem 1 relies on a result [B, 5.11] for which Ulam-nonmeasurability is essential. Nevertheless, we shall show in this note that Theorem 1 can be improved as follows (cf. [B, 5.14(c)]):

Theorem 2. *If* X *is locally compact and nonpseudocompact, then* X* *is not* Oz.

For the proof we require the two lemmas below. As noted in [vD_3, 20.3(1)], Lemma 2 is essentially due to Fine and Gillman (see the proof of [FG, 3.1]).

A space X is a P-*space* [GJ, 4J] (resp. an *almost*-P-*space* [Le]) if every zero-set of X is open (resp. regular closed) in X. It is easily seen that X is an almost-P-space if and only if every nonempty zero-set of X has nonempty interior [Le, 1.1].

Lemma 1. *(a) Every regular closed subset of an* Oz-*space is* Oz.

(b) X *is* Oz *if and only if the boundary of each regular closed subset of* X *is a zero-set in* X.

(c) An almost-P-*space is* Oz *if and only if it is extremally disconnected (in which case it is a* P-*space).*

(d) Every neighborhood retract of an Oz-*space is* Oz.

Proof. (a) is proved in [B, 5.3(a)], (b) follows readily from the fact that X is Oz if and only if every regular closed subset of X is a zero-set of X [B, 5.1], (c) follows from (b), and (d) follows from [B, 5.3(a)].

Lemma 2 (Fine and Gillman). *If* X *is locally compact and if* Z *is a nonempty zero-set of* βX *with* $Z \subset X^*$, *then* $\mathrm{int}_{X^*} Z \neq \emptyset$.

Proof of Theorem 2. Suppose, on the contrary, that X^* is Oz. Since X is nonpseudocompact, there is a (necessarily infinite [GJ, 9.5]) nonempty zero-set Z of βX with $Z \subset X^*$. Since Z is C^*-embedded in βX, each zero-set of Z is a zero-set of βX and is therefore, by Lemma 2, regular closed in X^* (and hence also in Z). In particular, Z is regular closed in

X^*, so Z is Oz by Lemma 1(a); and Z is an almost-P-space, so Z is a (compact) P-space by Lemma 1(c). But then Z is finite, a contradiction.

As an application of Theorem 2 we provide another proof of the following result essentially due to Comfort [C, 3.3]:

Corollary. If X^* *is an absolute neighborhood retract for compact spaces, then* X *is locally compact and pseudocompact.*

Proof. X is locally compact since X^* is compact. Moreover, X^* can be embedded as a neighborhood retract in a product Y of unit intervals, and by a result essentially due to Noble ([N], [B, 5.6]) Y is Oz. Hence X is pseudocompact by Lemma 1(d) and Theorem 2.

Remarks. (a) In an earlier version of this paper we based the proof of Theorem 2 on Theorem 1: If X is locally compact and nonpseudocompact, then N^* can be embedded as a neighborhood retract in X^* [vD_1, Lemma 1.1(c)], so X^* is not Oz by Lemma 1(d) and Theorem 1. The more direct proof given above was suggested by Eric van Douwen.

(b) By Theorem 2, N^* and R^* are not Oz. In [vD_4], van Douwen shows that Q^* is not Oz, and in [T_2] Terada shows that βR and βQ are not Oz.

(c) There exist locally compact pseudocompact spaces X for which X^* is Oz, and also for which X^* is not Oz (see [GJ, 9.K6]).

(d) In [C, 3.3], Comfort assumes that X^* is an absolute retract for compact spaces, but his proof obviously yields the corollary above as stated. In [vD_2], van Douwen proves that if X^* is a retract of βX, then X is pseudocompact. (This was originally proved by Comfort under CH [C, 2.6].) The following question is therefore suggested: Is X pseudocompact if X^* is a neighborhood retract of βX? Van Douwen has noted (oral communication) that as a consequence of [vD_2] the answer to this question is affirmative if X is locally compact. As an additional contribution, we remark that the answer is also affirmative if βX is Oz; the proof is omitted.

Added in proof, January 8, 1980: In a personal communication, van Douwen has shown that a slight modification of the proof of [vD_2, 0.1] answers the question above in the affirmative (with no restriction).

References

[B] R. L. Blair, *Spaces in which special sets are z-embedded*, Canad. J. Math. 28 (1976), 673-690.

[C] W. W. Comfort, *Retractions and other continuous maps from* βX *onto* $\beta X\setminus X$, Trans. Amer. Math. Soc. 114 (1965), 1-9.

[vD_1] E. K. van Douwen, *Transfer of information about* βN-N *via open remainder maps*, Illinois J. Math. (to appear).

[vD_2] ________, *Retractions from* βX *onto* βX-X, Gen. Top. Appl. 9 (1978), 169-173.

[vD_3] ________, *Remote points*, Dissertationes Math. (to appear).

[vD_4] ________, *The Čech-Stone remainder of some nowhere locally compact spaces* (manuscript).

[FG] N. J. Fine and L. Gillman, *Extension of continuous functions in* βN, Bull. Amer. Math. Soc. 66 (1960), 376-381.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

[La_1] E. P. Lane, *Insertion of a continuous function*, Topology Proceedings 4 (1979).

[La_2] ________, PM-*normality and the insertion of a continuous function*, Pacific J. Math. (to appear).

[Le] R. Levy, *Almost*-P-*spaces*, Canad. J. Math. 29 (1977), 284-288.

[N] N. Noble, C-*embedded subsets of products*, Proc. Amer. Math. Soc. 31 (1972), 613-614.

[T_1] T. Terada, *Note on* z-, C*-, *and* C-*embedding*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 13 (1975), 129-132.

[T_2] ________, *On spaces whose Stone-Čech compactification is* Oz (preprint).

Ohio University

Athens, Ohio 45701