REFINEMENTS OF LOCALLY COUNTABLE COLLECTIONS

by

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Several questions concerning spaces with a $\sigma$-locally countable base and paralindelöf spaces have proved to be surprisingly difficult. It is not known, for example, whether paralindelöf spaces must be paracompact or whether spaces with a $\sigma$-locally countable base must be screenable. Recent results appearing in [FR], and examples in [DGN] and [F], have contributed significantly to this area but many fundamental problems remain. Part of the reason for this seems to be that, in contrast with locally finite collections, there are only a small number of suitable techniques available for handling or refining locally countable collections. In this note, we give a result which allows for $\sigma$-closure preserving refinements of locally countable collections under certain conditions. By applying this theorem we obtain several new results, including the result that all regular $\theta$-refinable spaces with a $\sigma$-locally countable base are developable.

For convenience all regular spaces are assumed to be $T_1$ but, unless otherwise stated, no separation axioms are assumed. The set of natural numbers is denoted by $\mathbb{N}$. We begin immediately with the statement and proof of the main theorem; applications of this result and relationships to known results will be discussed later.
1. Theorem. If $\mathcal{P}$ is a collection of closed subsets of $X$ and $K$ is a point-finite open cover of $X$ such that each $K \in K$ intersects at most countably many elements of $\mathcal{P}$ then $\mathcal{P}$ has a $\sigma$-closure preserving refinement.

Proof. Assume $K = \{K(\alpha) : \alpha \in \Lambda\}$ where $\Lambda$ is well-ordered and $K(\alpha) \neq K(\beta)$ if $\alpha \neq \beta$. For each $\alpha \in \Lambda$ the set $H(\alpha) = \{P \in \mathcal{P} : P \cap K(\alpha) \neq \emptyset\}$ is countable, so express as $H(\alpha) = \{P(1,\alpha), P(2,\alpha), \ldots \}$.

(Make necessary adjustments in notation if $H(\alpha)$ is finite or empty.) For each $n \in \mathbb{N}$ let $F_n = \{x \in X : \text{ord}(x, K) \leq n\}$. For each finite sequence $(i_1, i_2, \ldots, i_n)$ of natural numbers and each $\beta \in \Lambda$, let $A(i_1, \ldots, i_n, \beta) = \{(\alpha_1, \ldots, \alpha_n) \in \Lambda^n : \alpha_1 < \alpha_2 < \cdots < \alpha_n = \beta \text{ and } P(i_1, \alpha_1) = P(i_2, \alpha_2) = \cdots = P(i_n, \alpha_n)\}$.

For each sequence $(i_1, \ldots, i_n) \in \mathbb{N}^n$, we will define a closure preserving collection $\partial(i_1, \ldots, i_n)$ - this will be done by induction on $n$.

Let $i \in \mathbb{N}$ (a sequence of length 1), and $\beta \in \Lambda$. Define $D(i, \beta) = F_1 \cap P(i, \beta) \cap K(\beta)$, and $E(i) = \{D(i, \beta) : \beta \in \Lambda\}$. Then $E(i)$ is a closure preserving collection (in fact, $E(i)$ is actually discrete). Now let $n \in \mathbb{N}$, $n > 1$ and assume that for any sequence $(j_1, \ldots, j_k) \in \mathbb{N}^k$, with $1 \leq k < n$, that $\partial(j_1, \ldots, j_k)$ is defined and is a closure preserving collection of subsets of $F_k$. For any $(i_1, \ldots, i_n) \in \mathbb{N}^n$, $\beta \in \Lambda$, define $E(i_1, \ldots, i_n, \beta) = \cup\{F_n \cap P(i_n, \beta) \cap K(\alpha_1) \cap \cdots \cap K(\alpha_n) : (\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)\}$. Then $H(i_1, \ldots, i_n, \beta) = \cup\{D(i_{j_1}, \ldots, i_{j_k}, \alpha_{j_k}) : (i_{j_1}, \ldots, i_{j_k}) \text{ is a subsequence of } (i_1, \ldots, i_n)\}$. 

To show $D(i, \ldots, i_n) = \{\beta \in \Lambda : \beta \in A\}$.

To show $D(i, \ldots, i_n)$ is closure preserving let $\Lambda' \subseteq \Lambda$ and suppose $x \in cl(U\{D(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$. If $x \in F_n - F_{n-1}$, then there exists $(\gamma_1, \ldots, \gamma_n) \in \Lambda^n$, with $\gamma_1 < \cdots < \gamma_n$ such that

$$x \in V = K(\gamma_1) \cap \cdots \cap K(\gamma_n).$$

Then $W \cap D(i_1, \ldots, i_n, \beta) = \emptyset$, for some $\beta \in \Lambda'$, implies $W \cap E(i_1, \ldots, i_n, \beta) = \emptyset$ (since $F_n \cap W \subseteq F_n - F_{n-1}$) which implies $(\gamma_1, \ldots, \gamma_n) \in A(i_1, \ldots, i_n, \beta)$ (so $\gamma_n = \beta$). This says there is only one $\beta \in \Lambda'$ such that $W \cap D(i_1, \ldots, i_n, \beta) = \emptyset$; it follows that $x \in cl(D(i_1, \ldots, i_n))$, for $\beta = \gamma_n$ and $\gamma_n \in \Lambda'$. Now suppose ord $(x, K) = k$, for $1 \leq k < n$; then there exists $(\gamma_1, \ldots, \gamma_k) \in \Lambda^k$, with $\gamma_1 < \cdots < \gamma_k$ such that $x \in V = K(\gamma_1) \cap \cdots \cap K(\gamma_k)$. If $x \in cl(U\{E(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$, then $x \in cl(D(i_{j_1}, \ldots, i_{j_r}, \alpha_{j_r}) \cap cl(D(i_1, \ldots, i_n, \beta))$ for some subsequence $(i_{j_1}, \ldots, i_{j_r})$ of $(i_1, \ldots, i_n)$ with $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$, since $D(i_{j_1}, \ldots, i_{j_r}, \alpha_{j_r}) = \beta \in \Lambda'$, $(i_{j_1}, \ldots, i_{j_r})$ is a subsequence of $(i_1, \ldots, i_n)$,

$$1 \leq r < n \text{ and } (\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$$

is closure preserving. Otherwise we have $x \in cl(U\{E(i_1, \ldots, i_n, \beta) : \beta \in \Lambda'\})$. Now note that

$V \cap E(i_1, \ldots, i_n, \beta) = \emptyset$, for some $\beta \in \Lambda'$, implies there is $(\alpha_1, \ldots, \alpha_n) \in A(i_1, \ldots, i_n, \beta)$ and a subsequence $(i_{j_1}, \ldots, i_{j_k})$ of $(i_1, \ldots, i_n)$ such that $\gamma_1 = \alpha_{j_1}, \gamma_2 = \alpha_{j_2}, \ldots, \gamma_k = \alpha_{j_k} \leq \beta$.

For every subsequence $(i_{j_1}, \ldots, i_{j_k})$ (of length $k$) of $(i_1, \ldots, i_n)$ let $\Lambda(i_{j_1}, \ldots, i_{j_k}) = \{\beta \in \Lambda' : \text{there is}}$
(a₁, · · · , aₙ) ∈ A(i₁, · · · , iₙ, β) such that

γ₁ = αj₁, · · · , γₖ = αjₖ.

Now, since there are only a finite number of such subsequences, there is some subsequence (i_{j₁}, · · · , i_{jₖ}) such that

x ∈ cl(∪{E(i₁, · · · , iₙ, β) : β ∈ A(i_{j₁}, · · · , i_{jₖ})}). For each

β ∈ A(i_{j₁}, · · · , i_{jₖ}) we have E(i₁, · · · , iₙ, β) ⊂ P(iₙ, β) = P(i_{jₖ}, γₖ). So x ∈ P(i_{jₖ}, γₖ) (since P(i_{jₖ}, γₖ) is closed)

and x ∈ F_{jₖ} ∩ K(γ₁) ∩ · · · ∩ K(γₖ); hence

x ∈ E(i_{j₁}, · · · , i_{jₖ}, γₖ) ⊂ D(i_{j₁}, · · · , i_{jₖ}, γₖ) = D(i₁, · · · , iₙ, β)

for any β ∈ A(i_{j₁}, · · · , i_{jₖ}). This shows D(i₁, · · · , iₙ) is closure preserving and D = ∪{D(i₁, · · · , iₙ) : n ∈ N,

(i₁, · · · , iₙ) ∈ N^n} is σ-closure preserving.

If D(i₁, · · · , iₙ, β) ∈ D, it follows by construction of D(i₁, · · · , iₙ, β) that

D(i₁, · · · , iₙ, β) ⊂ P(iₙ, β) ∈ P.

To complete the proof we need to show that D covers ∪P.

Let x ∈ ∪P and suppose ord (x, K) = n. There exist elements K(a₁), · · · , K(aₙ) of K such that x ∈ K(a₁) ∩ · · · ∩ K(aₙ) and

a₁ < a₂ < · · · < aₙ. For each j, 1 ≤ j ≤ n, there is i_j ∈ N

so that x ∈ P(i_{j₁}, a_j) ∈ H(a_j) and P(i₁, a₁) = P(i₂, a₂) = · · · = P(iₙ, aₙ). It follows that x ∈ D(i₁, · · · , iₙ, aₙ) ∈ D and the theorem is proved.

A direct application of Theorem 1 shows that in a meta-compact space X any locally countable collection of closed sets has a σ-closure preserving refinement. A little more work gives a sharpened version of this in θ-refinable spaces. Recall that a space X is θ-refinable [WoW] if for any open cover U of X there is a sequence {S_n}^∞_n=1 of open covers of X,
each refining $U$, such that for any $x \in X$ there is $n \in \mathbb{N}$ where $0 < \text{ord}(x, \mathcal{C}_n) < \omega$. The sequence $\{\mathcal{C}_n\}_{n=1}^{\infty}$ is called a $\theta$-refinement of $U$. If, in the above definition, the collections $\mathcal{C}_n$ are not required to cover $X$, then $X$ is said to be weakly $\theta$-refinable [BL].

2. Corollary. In a $\theta$-refinable space $X$ any locally countable collection of closed subsets has a $\sigma$-closure preserving refinement. Hence every $\sigma$-locally countable closed collection has a $\sigma$-closure preserving (closed) refinement.

Proof. Suppose $\mathcal{P}$ is a locally countable collection of closed subsets of $X$. There is an open cover $U$ of $X$ such that each $U \in U$ intersects at most countably many elements of $\mathcal{P}$. Let $\{\mathcal{C}_n\}_{n=1}^{\infty}$ be a $\theta$-refinement of $U$. For each $n,k \in \mathbb{N}$, let

$$Y_{n,k} = \{x \in X : \text{ord}(x, \mathcal{C}_n) \leq k\},$$

$$K_{n,k} = \{G \cap Y_{n,k} : G \in \mathcal{C}_n\},$$

and

$$P_{n,k} = \{P \cap Y_{n,k} : P \in \mathcal{P}\}.$$

By applying Theorem 1 to the space $Y_{n,k}$ it follows that $P_{n,k}$ has a $\sigma$-closure preserving refinement $\partial_{n,k}$ (relative to $Y_{n,k}$), and since $Y_{n,k}$ is closed in $X$ it follows that $\partial = \bigcup \partial_{n,k} : n,k \in \mathbb{N}$ is a $\sigma$-closure preserving refinement of $\mathcal{P}$. That completes the proof.

It is expected that some sort of covering property (such as $\theta$-refinable) would be necessary in Corollary 2. This is illustrated by Example 3 and Example 4 below. Example 3 is very simple and shows that locally countable covers need not have any "nice refinements". Example 4, due to G. Gruenhage, is described in [DGN] and shows that
the \( \theta \)-refinable condition cannot be weakened to weakly \( \theta \)-refinable in Corollary 2 (and Corollaries 5, 6, and 7 below).

3. Example. There is a completely regular space \( X \) with a locally countable cover \( \mathcal{U} \) of open and closed sets such that \( \mathcal{U} \) does not have a \( \sigma \)-closure preserving refinement.

Proof. Let \( X = \{ (\alpha, \beta) \in \omega_1 \times \omega_1 : \alpha < \beta \} \) with the relative topology inherited from \( \omega_1 \times \omega_1 \). For each \( \alpha \in \omega_1 \), let

\[
U_\alpha = [0, \alpha] \times (\alpha, \omega_1).
\]

Then each \( U_\alpha \) is open and closed, and the collection \( \mathcal{U} = \{ U : \alpha < \omega_1 \} \) is a locally countable cover of \( X \) with no \( \sigma \)-closure preserving refinement. The details are left to the reader.

4. Example. There is an example of a completely regular nondevelopable space \( Z \) with a \( \sigma \)-locally countable base. This space is screenable (hence weakly \( \theta \)-refinable) but not \( \theta \)-refinable.

See Example 3.3 in [DGN] for the details. The following properties (with some repetition) of \( Z \) are easily verified.

(a) \( Z \) is weakly \( \theta \)-refinable and every open cover of \( Z \) has a \( \sigma \)-locally countable refinement but \( Z \) is not subparacompact.

(b) \( Z \) is weakly \( \sigma \)-refinable and has a \( \sigma \)-locally countable network but \( Z \) is not a \( \sigma \)-space.

(c) \( Z \) is weakly \( \theta \)-refinable and has a \( \sigma \)-locally countable base but \( Z \) is not developable.

The above statements (a), (b), and (c) should be con-
trasted with Corollaries 5, 6, and 7 below in order to see that these results cannot be significantly improved by weakening the \( \theta \)-refinable condition. We view Corollary 7 as the main result in this group; this extends results given by Fleissner and Reed in [FR] where it was shown that a regular space \( X \), with a \( \sigma \)-locally countable base, is developable if \( X \) is subparacompact or if (under Martin's Axiom) \( X \) is metacompact with \( |X| < c \).

5. Corollary. If \( X \) is a regular \( \theta \)-refinable space in which every open cover has a \( \sigma \)-locally countable refinement then \( X \) is subparacompact.

Proof. Using regularity we see that every open cover of \( X \) has a \( \sigma \)-locally countable closed refinement and by Corollary 2 it follows that every open cover of \( X \) has a \( \sigma \)-closure preserving closed refinement. Theorem 1.2 in [Bu] shows that \( X \) is subparacompact.

6. Corollary. A regular \( \theta \)-refinable space \( X \) with a \( \sigma \)-locally countable network is a \( \sigma \)-space.

Proof. It follows from the proofs of Theorem 1 and Corollary 5 that a \( \sigma \)-locally countable closed network for \( X \) can be replaced by a \( \sigma \)-closure preserving closed network. According to [NS] this is equivalent to \( X \) being a \( \sigma \)-space.

7. Corollary. A regular \( \theta \)-refinable space \( X \) with a \( \sigma \)-locally countable base is a Moore space.

Proof. Corollary 5 shows that \( X \) is subparacompact; Fleissner and Reed [FR] have shown that a regular subparacompact space with a \( \sigma \)-locally countable base is a Moore space.
Corollary 7 can be viewed as a companion to Fedorčuk's result [Fe] that a paracompact space with a $\sigma$-locally countable base is metrizable. C. Aull has also given results [A] where covering properties convert to corresponding base properties in a space with a $\sigma$-locally countable base.

A space $X$ is said to be paralindelöf if every open cover of $X$ has a locally countable open refinement. Notice that Corollary 5 says that a regular $\theta$-refinable paralindelöf space $X$ must be subparacompact. It is not known whether paralindelöf spaces must always be subparacompact.

We conclude with a simple result related to the question of whether a space with a $\sigma$-locally countable base is screenable (or equivalently has a $\sigma$-disjoint base). This proof indicates that if a space $X$, with a $\sigma$-locally countable base $\beta$, has a $\sigma$-disjoint base then a $\sigma$-disjoint base for $X$ can be found by using only unions of elements of $\beta$ (as opposed to intersections, differences, etc.).

8. Proposition. If a space $X$ has a $\sigma$-locally countable base $\beta$ and a $\sigma$-disjoint base $\mathcal{D}$ then $X$ has a base $\mathcal{G}$ which is simultaneously $\sigma$-locally countable and $\sigma$-disjoint.

Proof. Suppose $\beta = \bigcup_{n=1}^{\infty} \beta_n$ where each $\beta_n$ is locally countable and $\mathcal{D} = \bigcup_{n=1}^{\infty} D_n$ where each $D_n$ is a disjoint collection. For any $n,k \in N$ and $D \in D_n$, let

$$G(D,n,k) = \{B: B \in \beta_k, B \subseteq D\}$$

$$\mathcal{G}(n,k) = \{G(D,n,k): D \in D_n\}.$$  

It is easily shown that each $\mathcal{G}(n,k)$ is simultaneously a locally countable and disjoint collection, and

$$\mathcal{G} = \bigcup_{n,k \in N} \mathcal{G}(n,k)$$  

is a base for $X$.  

References


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