TOPOLOGY PROCEEDINGS

Volume 4, 1979

Pages 51–65

http://topology.auburn.edu/tp/

PRODUCTS OF SPACES WITH PROPERTIES OF PSEUDO-COMPACTNESS TYPE

by

W. W. Comfort

Topology Proceedings

Web:	http://topology.auburn.edu/tp/
Mail:	Topology Proceedings
	Department of Mathematics & Statistics
	Auburn University, Alabama 36849, USA
E-mail:	topolog@auburn.edu
ISSN:	0146-4124

PRODUCTS OF SPACES WITH PROPERTIES OF PSEUDO-COMPACTNESS TYPE

W. W. Comfort[1]

1. Introduction

When is the product of pseudo-(α,β)-compact spaces a pseudo-(α,β)-compact space? We show: For regular, uncountable α a product is pseudo-(α,β)-compact if and only if each finite subproduct is; for $\alpha = \beta$ with $\mathrm{cf}(\alpha) > \omega$ the corresponding statement holds (a result of Argyros); for $\alpha = \beta$ with $\mathrm{cf}(\alpha) = \omega$ the corresponding statement fails; for α singular with $\alpha > \beta$ the corresponding statement fails; and for $0 < m < \omega$ there is a space X such that X^{m-1} is pseudo-(α,β)-compact and X^m is not.

These results settle most, but perhaps not all, of the questions which arise naturally concerning the behavior of pseudo-(α,β)-compactness under the formation of products. Some related open questions are stated explicitly.

This investigation was undertaken in connection with an extended work [5] co-authored with S. Negrepontis. For complete and detailed proofs of the results, for historical and bibliographical references, and for similar (and different) results concerning other chain conditions in topology, the reader should consult [5].

2. Definitions, Notation and Conventions

The spaces we consider are not required or assumed to satisfy any special separation axioms. It will be noted, however, that the spaces we construct or define (as examples of spaces with particular properties) are completely regular, Hausdorff spaces.

The symbols α, β, κ and λ denote cardinal numbers; ω is the least infinite cardinal number. The symbols ξ and η denote ordinal numbers.

Definition. Let X be a space.

(a) A family $\{U_\xi : \xi < \alpha\}$ of subsets of X is <u>locally</u> $< \beta$ (in X) if for every element x of X there is a neighborhood V of x such that

$$|\{\xi < \alpha : V \cap U_\xi \neq \emptyset\}| < \beta.$$

(b) The space X is <u>pseudo</u>-(α,β)-<u>compact</u> if no set of non-empty open subsets of X indexed by α is locally $< \beta$.

We note that according to this use of terminology, a space X is pseudo-(ω,ω)-compact if and only if each locally finite family of non-empty, open subsets of X is finite.

Notation. The cardinal α is <u>strongly</u> κ-<u>inaccessible</u> if

(a) $\kappa < \alpha$ and

(b) if $\beta < \alpha$ and $\lambda < \kappa$, then $\beta^\lambda < \alpha$.

When α is strongly κ-inaccessible, we write $\kappa \ll \alpha$.

We note that $\alpha^+ \ll (2^\alpha)^+$ for all infinite cardinals α.

Definition. Let $\kappa \geq \omega$ and let $\{X_i : i \in I\}$ be a set of spaces. The κ-<u>box</u> <u>topology</u> on $\Pi_{i\in I}X_i$ is the topology generated by sets of the form $\Pi_{i\in I}U_i$ with U_i open in X_i and with $|\{i \in I: U_i \neq X_i\}| < \kappa$.

We set $X_I = \Pi_{i\in I}X_i$; the set X_I with the κ-box topology is denoted $(X_I)_\kappa$.

We note that the ω-box topology is the usual product topology; and if $|I| < \kappa$ then the κ-box topology on X_I is the box topology.

3. Pseudo-Compactness Properties of Large Products

We show that under appropriate conditions the question whether a product $(X_I)_\kappa$ is pseudo-(α,β)-compact is determined by the behavior of "small" subproducts.

3.1. *Theorem. Let* $\alpha \geq \beta \geq \kappa$ *and* $\omega \leq \kappa \ll \alpha$ *with* α *regular, and let* $\{X_i: i \in I\}$ *be a set of non-empty spaces. The following statements are equivalent.*

(a) $(X_I)_\kappa$ *is pseudo-*(α,β)*-compact;*

(b) $(X_J)_\kappa$ *is pseudo-*(α,β)*-compact for all* $J \subset I$ *with* $|J| < \kappa$.

Proof. (a) $\Rightarrow$ (b). For $J \neq \emptyset$ the space $(X_J)_\kappa$ is the continuous image, under the projection function, of the space $(X_I)_\kappa$; hence $(X_J)_\kappa$ is pseudo-(α,β)-compact.

(b) $\Rightarrow$ (a). We show that no family $\{U^\xi: \xi < \alpha\}$ of non-empty, basic open subsets of $(X_I)_\kappa$ is locally $< \beta$ in $(X_I)_\kappa$.

For $\xi < \alpha$ there are U_i^ξ open in X_i such that $U^\xi = \Pi_{i \in I} U_i^\xi$ and such that with $R(\xi) = \{i \in I: U_i^\xi \neq X_i\}$ we have $|R(\xi)| < \kappa$. From the Erdös-Rado theorem on quasi-disjoint families (see for example [7], [4] (Theorem 3.2) or [5] (Theorem 1.3)) there are a subset A of α with $|A| = \alpha$ and a subset J of I such that

$$R(\xi) \cap R(\xi') = J \text{ whenever } \xi, \xi' \in A \text{ and } \xi \neq \xi'.$$

We consider two cases.

Case 1. $J = \emptyset$. For $i \in I$ there is at most one $\xi \in A$ such that $i \in R(\xi)$. We define $x \in X_I$ by the rule

$$x_i \in U_i^\xi \text{ if } i \in R(\xi) \quad (\xi \in A)$$
$$\in X_i \text{ if } i \in I\backslash \cup_{\xi \in A} R(\xi).$$

It is then clear that $x \in \cap_{\xi \in A} U^\xi$, so that no neighborhood of x hits fewer than β of the sets U^ξ $(\xi \in A)$.

Case 2. $J \neq \emptyset$. We have $|J| < \kappa$. Since $(X_J)_\kappa$ is pseudo-(α,β)-compact and $\pi_J[U^\xi]$ is open in $(X_J)_\kappa$ for all $\xi \in A$ there is $p \in (X_J)_\kappa$ such that for every neighborhood V_J of p in $(X_J)_\kappa$ we have

$$|\{\xi \in A: V_J \cap \pi_J[U^\xi] \neq \emptyset\}| \geq \beta.$$

For $i \in I\backslash J$ there is at most one $\xi \in A$ such that $i \in R(\xi)$. We define $x \in (X_I)_\kappa$ by the rule

$$x_J = p,$$
$$x_i \in U_i \text{ if } i \in I\backslash J,\ i \in R(\xi),\ \xi \in A$$
$$\in X_i \text{ if } i \in I\backslash (J \cup \cup_{\xi \in A} R(\xi)).$$

We claim that if V is a basic neighborhood of x in $(X_I)_\kappa$ then

$$|\{\xi \in A: V \cap U^\xi \neq \emptyset\}| \geq \beta.$$

We set

$$B = \{\xi \in A: \pi_J[U] \cap \pi_J[U^\xi] \neq \emptyset\}, \text{ and}$$
$$B(i) = \{\xi \in B: V_i \cap U_i^\xi = \emptyset\} \text{ for } i \in R(V)\backslash J.$$

With $R(V) = \{i \in I: V_i \neq X_i\}$ we have $|B(i)| \leq 1$ for $i \in R(V)\backslash J$ and hence

$$|\{\xi \in B: V \cap U^\xi = \emptyset\}| \leq |R(V)| < \kappa.$$

Thus

$$|\{\xi \in B: V \cap U^\xi \neq \emptyset\}| \geq \beta,$$

as required.

A special case of Theorem 3.1, and several related statements, are given in [3].

3.2. *Corollary. Let* $\alpha \geq \beta \geq \omega$ *and* $\alpha > \omega$ *with* α *regular, and let* $\{X_i : i \in I\}$ *be a set of non-empty spaces. Then* X_I *is pseudo-*(α,β)*-compact if and only if* X_J *is pseudo-*(α,β)*-compact for all finite* $J \subset I$.

An attempt to prove a statement analogous to Theorem 3.1 for singular cardinals α founders on the fact that the Erdös-Rado Theorem is available in the form used above only for regular α. Various mathematicians, however, have proved statements for singular cardinals similar to (but more complicated than) the Erdös-Rado Theorem. A preliminary version was given by Noble and Ulmer [13]; a strong version for the case $\kappa = \omega$ was given by Shelah [14]; and a very satisfactory version for general κ, which is the basis for establishing a number of chain conditions in κ-box products, appears in the doctoral dissertation of Argyros [1]. For a careful statement and proof of the generalization to singular cardinals of the Erdös-Rado Theorem, as well as for a proof of the following theorem (due also to Argyros), the reader is referred to [5].

3.3. *Theorem. Let* $\omega \leq \kappa \ll \alpha$ *and* $\kappa \ll \mathrm{cf}(\alpha)$ *and let* $\{X_i : i \in I\}$ *be a set of non-empty spaces. The following statements are equivalent.*

(a) $(X_I)_\kappa$ *is pseudo-*(α,α)*-compact;*

(b) $(X_J)_\kappa$ *is pseudo-*(α,α)*-compact for all* $J \subset I$ *with* $|J| < \kappa$.

3.4. *Corollary. Let* α *be an infinite cardinal with* $\mathrm{cf}(\alpha) > \omega$*, and let* $\{X_i : i \in I\}$ *be a set of non-empty spaces.*

Then X_I *is pseudo-*(α,α)*-compact if and only if* X_J *is pseudo-*(α,α)*-compact for all finite* $J \subset I$.

These results give no information for $\alpha > \beta$ with α singular. The following example, communicated in correspondence by Eric van Douwen and included here with his kind permission, settles the problem under consideration for all such pairs of cardinals.

3.5. *Theorem. Let* α *be an (infinite) singular cardinal,* $\omega \leq \kappa \ll \alpha$ *and* $\kappa \leq \mathrm{cf}(\alpha)$. *There is a set* $\{X_i : i \in I\}$ *of completely regular, Hausdorff spaces such that*

(i) $(X_J)_\kappa$ *is pseudo-*(α,β)*-compact for all* $J \subset I$, $|J| < \kappa$, $2 \leq \beta < \alpha$; *and*

(ii) $(X_I)_\kappa$ *is not pseudo-*(α,β)*-compact if* $2 \leq \beta < \alpha$.

Proof. There is a set $\{\alpha_\sigma : \sigma < \mathrm{cf}(\alpha)\}$ of cardinal numbers such that

$$\alpha_0 = 0,$$
$$\alpha_{\sigma'} < \alpha_\sigma < \alpha \text{ for } \sigma' < \sigma < \mathrm{cf}(\alpha), \text{ and}$$
$$\Sigma_{\sigma<\mathrm{cf}(\alpha)} \alpha_\sigma = \alpha.$$

We set $I = \mathrm{cf}(\alpha)$ and for $\sigma \in I$ we denote by X_σ the (discrete) space α_σ. We verify (i) and (ii).

(i) Let $J \subset I$ with $|J| < \kappa$ and set $\gamma = |(X_J)_\kappa|$. Since $\kappa \ll \alpha$ and X_J is the product of fewer than κ sets, each of cardinality $< \alpha$, we have from $\kappa \leq \mathrm{cf}(\alpha)$ that $\gamma < \alpha$. Let $\beta < \alpha$, suppose that $\{U_\xi : \xi < \alpha\}$ is a locally $< \beta$ family of non-empty subsets of $(X_J)_\kappa$, and set

$$A(x) = \{\xi < \alpha : x \in U_\xi\} \text{ for } x \in (X_J)_\kappa.$$

Then $|A(x)| < \beta$ for $x \in (X_J)_\kappa$ and since $\cup_x A(x) = \alpha$ we have

$$\alpha \le \Sigma\{|A(x)|: x \in (X_J)_\kappa\} \le \beta \cdot \gamma < \alpha,$$

a contradiction.

(ii) We set

$$U(0) = \{x \in (X_I)_\kappa: x_0 = 0\}, \text{ and}$$
$$U(\sigma,\eta) = \{x \in (X_I)_\kappa: x_0 = \sigma, x_\sigma = \eta\} \text{ for}$$
$$0 < \sigma < cf(\alpha), \eta < \alpha_\sigma.$$

The family

$$\{U(0)\} \cup \{U(\sigma,\eta): 0 < \sigma < cf(\alpha), \eta < \alpha_\sigma\}$$

is an open cover of $(X_J)_\kappa$, of cardinality α, by pairwise disjoint, non-empty open sets. Thus $(X_J)_\kappa$ is not pseudo-$(\alpha,2)$-compact.

3.6. *Corollary. Let* α *be an (infinite) singular cardinal. There is a set* $\{X_i: i \in I\}$ *of completely regular, Hausdorff spaces such that* X_J *is pseudo-*(α,β)*-compact for all finite* $J \subset I$, *all* $\beta < \alpha$, *and* X_I *is not pseudo-*(α,β)*-compact if* $2 \le \beta < \alpha$.

The device used in the proof of Theorem 3.5 to define a cover by pairwise disjoint open subsets of a product of discrete spaces has served related purposes in the past; see for example Mycielski [12] and Engelking [6] (Exercise 3.2.F(b)).

It is natural to wonder whether spaces X_i with the properties of those in 3.5 and 3.6 might be chosen pairwise homeomorphic. That such a choice is not possible follows from an argument shown to me by Negrepontis and developed by Argyros to furnish an alternative, and quite elegant, proof of a product-space theorem of Shelah [14] concerning calibres. Here is the precise statement.

3.7. *Theorem. Let* $\alpha \geq \beta$ *and* $\mathrm{cf}(\alpha) > \omega$. *If* X *is a space and* X^n *is pseudo-*(α,β)*-compact for all* $n < \omega$, *then* X^I *is pseudo-*(α,β)*-compact for all* I.

A detailed proof is given in [5].

3.8. *Question.* Is the analogue for κ-box topologies of Theorem 3.7 a true statement?

In connection with κ-box topologies with $\kappa > \omega$, it is well to recognize that pseudo-compactness properties are not always achieved by small products. The following simple theorem will illustrate.

3.9. *Theorem. Let* κ *and* λ *be cardinal numbers with* $\kappa > \lambda \geq \omega$ *and let* $\{X_i\colon i \in I\}$ *be a set of regular, Hausdorff spaces with* $|I| \geq \lambda$ *and with* $|X_i| > 1$ *for* $i \in I$. *Then* $(X_I)_\kappa$ *is not pseudo-*$(2^\lambda,2)$*-compact.*

Proof. It is enough to treat the case $I = \lambda$. For $\eta < \lambda$ there are non-empty, open subsets U_η, V_η of X_η such that $\overline{U}_\eta$ and $\overline{V}_\eta$ have disjoint neighborhoods in X_η. For $A \subset \lambda$ we set

$$W(A) = (\cap_{\eta\in A}\pi_\eta^{-1}(U_\eta)) \cap (\cap_{\eta\in\lambda\setminus A}\pi_\eta^{-1}(V_\eta));$$

then for $x \in (\Pi_{\eta<\lambda}X_\eta)_\kappa$ there is a neighborhood W of x such that

$$|\{A \subset \lambda\colon W \cap W(A) \neq \emptyset\}| \leq 1.$$

3.10. *Corollary. Let* α, κ *and* λ *be infinite cardinals with* $\kappa > \lambda$ *and let* $\{X_i\colon i \in I\}$ *be a set of regular, Hausdorff spaces with* $|I| \geq \lambda$ *and with* $|X_i| > 1$ *for* $i \in I$.

(a) If $2 \leq \beta \leq \alpha \leq 2^\lambda$ *then* $(X_I)_\kappa$ *is not pseudo-*(α,β)*-*

compact.

(b) If $\mathrm{cf}(\alpha) \leq 2^{\lambda}$ *then* $(X_I)_\kappa$ *is not pseudo-*(α,α)*-compact.*

4. Finite Products

Some Examples. The results of §3 make it clear that questions in product spaces concerning properties of pseudo-compactness type revert in many cases to a consideration of small subproducts. It then becomes appropriate to ask: To what extent are these properties finitely productive? We here describe some examples, using the usual product topology, indicating some conditions under which small products retain the properties in question and others in which they do not.

Again it is convenient to consider separately the cases of regular and of singular cardinals; the emphasis in this connection shifts, however, to the second cardinal in question (i.e., to the "β" of property pseudo-(α,β)-compact).

Definition. For $\alpha \geq \omega$ and $f \in \alpha^{\alpha}$ we denote by $\overline{f}$ the Stone extension of f; that is, $\overline{f}$ is that (unique) continuous function from the Stone-Čech compactification $\beta(\alpha)$ of the discrete space α to $\beta(\alpha)$ such that $\overline{f}|\alpha = f$.

Notation. Let $\alpha \geq \omega$ and $p \in \beta(\alpha)$. Then

(a) the type of p in $\beta(\alpha)$, denoted T(p), is the set $T(p) = \{\overline{f}(p)\colon f \text{ is a permutation of } \alpha\}$;

(b) the norm of p, denoted $\|p\|$, is the cardinal number $\|p\| = \min\{|A|\colon A \in p\}$; and

(c) $N_\kappa(\alpha) = \{p \in \beta(\alpha)\colon \|p\| < \kappa\}$.

The spaces T(p) for $p \in \beta(\alpha)$ were introduced by Frolík [8] in connection with his proofs (see Frolík [8], [10]) of the statement, given in ZFC without appealing to the continuum hypothesis or to any special set-theoretic assumption, that the space $\beta(\omega)\backslash\omega$ is not homogeneous.

We note that $N_\kappa(\alpha) = \cup\{cl_{\beta(\alpha)}A\colon A \subset \alpha, |A| < \kappa\}$.

We note also that if $\kappa \geq \omega$ then $\alpha \subset N_\kappa(\alpha) \subset \beta(\alpha)$.

4.1. *Lemma. Let* $\alpha \geq \beta \geq \omega$, $p \in \beta(\alpha)$ *with* $||p|| = \beta$, *and* $0 < m < \omega$. *For* $k < m$ *let* X_k *be a space such that*

(a) $\alpha \cup T(p) \subset X_k \subset \beta(\alpha)$ *if* β *is regular, and*

(b) $N_{cf(\beta)^+}(\alpha) \cup T(p) \subset X_k \subset \beta(\alpha)$ *if* β *is singular;*

and set $X = \Pi_{k<m}X_k$.

Then X *is pseudo-*(β,β)*-compact.*

The proof proceeds by induction on m and the fact that if $s\colon \beta \to X$ satisfies

$|(\pi_k \circ s)[B]| = \beta$ for all $B \subset \beta$ with $|B| = \beta$ and $k < m$,

then there is $q \in X$ such that $|\{\eta < \beta\colon s(\eta) \in V\}| = \beta$ for every neighborhood V of q. Details are given in 9.3-9.5 of [5].

It was noted by Glicksberg [11] in 1959 that a product of spaces is pseudo-(ω,ω)-compact if and only if each countable subproduct is. Reading this result I asked at the annual meeting of the American Mathematical Society in January, 1966 whether in this statement the word "countable" could legitimately be replaced by "finite"; I asked also whether, given spaces X, Y and Z with $X \times Y$, $X \times Z$ and $Y \times Z$ all pseudo-(ω,ω)-compact, the space $X \times Y \times Z$ must also be pseudo-(ω,ω)-compact. Frolík and I in 1967 independently contributed negative answers to these questions ([9], [2]).

Using Lemma 4.1 we see now that with little change our constructions can be adapted to give more general results.

It is clear from the definitions that if $\alpha \geq \alpha' \geq \beta' \geq \beta$, then every pseudo-$(\alpha',\beta')$-compact space is pseudo-(α,β)-compact. Thus within the hierarchy of properties of pseudo-compactness type, pseudo-(β,β)-compactness is a strong property. It is reasonable to ask whether finite products of spaces with this property also enjoy it and, if not, how bad or weak such a product may become with respect to properties of pseudo-compactness type. The following theorem indicates that for $\beta \geq \omega$, there is for every $\alpha \geq \beta$ a finite set of pseudo-(β,β)-compact spaces whose product is not even pseudo-(α,ω)-compact.

4.2. *Theorem. Let* $\alpha \geq \beta \geq \omega$ *and* $1 < m < \omega$.

(a) If β *is regular there is a set* $\{X_k\colon k < m\}$ *of completely regular, Hausdorff spaces such that*

$\Pi_{k\in A}X_k$ *is pseudo-*(β,β)*-compact whenever* $A \subset m$, $A \neq m$, *and*

$\Pi_{k<m}X_k$ *is not pseudo-*(α,ω)*-compact.*

(b) If β *is singular there is a set* $\{X_k\colon k < m\}$ *of completely regular, Hausdorff spaces such that*

$\Pi_{k\in A}X_k$ *is pseudo-*(β,β)*-compact whenever* $A \subset m$, $A \neq m$, *and*

$\Pi_{k<m}X_k$ *is not pseudo-*$(\alpha,(\mathrm{cf}(\beta))^+)$*-compact.*

Proof. There is a subset $\{p_n\colon n < m\}$ of $\beta(\alpha)$ such that

$||p_n|| = \beta$ for $n < m$, and

$T(p_n) \cap T(p_{n'}) = \emptyset$ for $n < n' < m$.

For $n < m$ we set

$$Y_n = \alpha \cup T(p_n) \text{ in (a), and}$$

$$Y_n = N_{(cf(\beta))^+}(\alpha) \cup T(p_n) \text{ in (b),}$$

and we set

$$X_k = \cup\{Y_n : n < m,\ n \neq k\} \text{ for } k < m.$$

For $A \subset m$ with $A \neq m$ there is $n \in m\backslash A$ and for $k \in A$ we have $Y_n \subset X_k$; hence $\Pi_{k\in A}X_k$ contains as a dense subspace the space $Y_n^{|A|}$. It follows from Lemma 4.1 that X is pseudo-(β,β)-compact.

In (a), the space $\Pi_{k<m}X_k$ contains a copy of the discrete space α as an open-and-closed subspace; hence $\Pi_{k<m}X_k$ is not pseudo-(α,ω)-compact.

In (b), the space $\Pi_{k<m}X_k$ contains a copy of the space $N_{(cf(\beta))^+}(\alpha)$ as an open-and-closed subspace; hence $\Pi_{k<m}X_k$ is not pseudo-$(\alpha,(cf(\beta))^+)$-compact.

The spaces X_k of Theorem 4.2 can be amalgamated, as follows.

4.3. *Corollary. Let* $\alpha \geq \beta \geq \omega$ *and* $1 < m < \omega$.

(a) If β *is regular there is a completely regular, Hausdorff space* X *such that* X^{m-1} *is pseudo-*(β,β)*-compact, and* X^m *is not pseudo-*(α,ω)*-compact.*

(b) If β *is singular there is a completely regular, Hausdorff space* X *such that* X^{m-1} *is pseudo-*(β,β)*-compact and* X^m *is not pseudo-*$(\alpha,(cf(\beta))^+)$*-compact.*

Proof. Take for X the topological (disjoint) union of the spaces X_k $(k < m)$ defined in the proof of Theorem 4.2.

4.4. *Corollary. Let* $\alpha \geq \beta \geq \omega$ *and* $1 < m < \omega$. *There is a completely regular, Hausdorff space* X *such that* X^{m-1}

is pseudo-(α,β)*-compact and* X^m *is not pseudo-*(α,β)*-compact.*

Proof. This follows from Corollary 4.3(a) for β regular, and from Corollary 4.3(b) for β singular.

The following question, though of only marginal interest and importance, calls attention to the fact that the statements of parts (a) and (b) of Theorem 4.2, though similar, are not identical.

4.5. *Question.* Let $\alpha \geq \beta \geq \omega$ with β singular and let $1 < m < \omega$. Is there a (completely regular, Hausdorff) space X such that X^{m-1} is pseudo-(β,β)-compact and X^m is not pseudo-(α,ω)-compact?

We have noted above (Corollary 3.4) that if $cf(\alpha) > \omega$ and $\{X_i: i \in I\}$ is a set of spaces for which every finite subproduct is pseudo-(α,α)-compact, then $\Pi_{i\in I}X_i$ is pseudo-(α,α)-compact. We see next that for $cf(\alpha) = \omega$ the analogous statement fails. In fact, a single example to this end can be chosen simultaneously suitable for all α with $cf(\alpha) = \omega$.

4.6. *Theorem. There is a completely regular, Hausdorff space* X *such that* X^m *is pseudo-*(α,α)*-compact for all* α *such that* $cf(\alpha) = \omega$ *and all* $m < \omega$, *and* X^ω *is not pseudo-*(α,α)*-compact for any* α *such that* $cf(\alpha) = \omega$.

Proof. There is a subset $\{p_n: n < \omega\}$ of $\beta(\omega)$ such that

$$||p_n|| = \omega \text{ for } n < \omega, \text{ and}$$

$$T(p_n) \cap T(p_{n'}) = \emptyset \text{ for } n < n' < \omega.$$

We choose $\{A_n: n < \omega\}$ such that

$$\cup_{n<\omega}A_n = \omega,$$

$$|A_n| = \omega \text{ for } n < \omega, \text{ and}$$

$$A_n \cap A_{n'} = \emptyset \text{ for } n < n' < \omega,$$

and we set

$$B_n = A_n \cup \bigcup\{(\mathrm{cl}_{\beta(\omega)} A_n) \cap T(p_k) : k < \omega,\ k \neq n\},$$

$$C = \beta(\omega) \setminus \bigcup\{\mathrm{cl}_{\beta(\omega)} A_n : n < \omega\}, \text{ and}$$

$$X = C \cup \bigcup\{B_n : n < \omega\}.$$

The reader who does not wish to verify for himself that the space X is as required is referred to [5] (Theorem 9.10) for the details.

We note that for X defined in the proof of Theorem 4.6 the space X^I is pseudo-(α,ω)-compact for all sets I and all cardinal numbers $\alpha > \omega$. Indeed for $J \subset I$ with $|J| < \omega$ the space X^J is pseudo-(ω,ω)-compact and hence pseudo-(ω^+,ω)-compact; thus X^I is pseudo-(ω^+,ω)-compact (by Corollary 3.2) and hence pseudo-(α,ω)-compact for all $\alpha > \omega$.

References

1. S. Argyros, *Isomorphic embeddings of $\ell'(\Gamma)$ into classical Banach spaces*, Doctoral dissertation, Athens University, 1977. [In Greek.]
2. W. W. Comfort, *A nonpseudocompact product space whose finite subproducts are pseudocompact*, Math. Annalen 170 (1967), 41-44.
3. ________ and S. Negrepontis, *Continuous functions on products with strong topologies*. In: General Topology and its Relations to Modern Analysis and Algebra 3, Proc. Third (1971) Prague Topological Symposium, Academia Prague, Prague, 1972, 89-92.
4. ________, *The Theory of Ultrafilters*, Grundlehren der math. Wissenschaften vol. 211, Springer-Verlag, Berlin-Heidelberg-New York, 1974.
5. ________, *Chain Conditions in Topology*, Cambridge

University Press (to appear).

6. R. Engelking, *Outline of General Topology*, North-Holland Publishing Co., Amsterdam. PWN - Polish Scientific Publishers, Warszawa, 1968.
7. P. Erdös and R. Rado, *Intersection theorems for systems of sets (II)*, J. London Math. Soc. 44 (1969), 467-479.
8. Z. Frolík, *Sums of ultrafilters*, Bull. Amer. Math. Soc. 73 (1967), 87-91.
9. ________, *On two problems of W. W. Comfort*, Commentationes Math. Univ. Carolinae 8 (1967), 139-144.
10. ________, *Fixed points of maps of* βN, Bull. Amer. Math. Soc. 74 (1968), 187-191.
11. Irving Glicksberg, *Stone-Čech compactifications of products*, Trans. Amer. Math. Soc. 90 (1959), 360-382.
12. Jan Mycielski, *α-incompactness of* N^{α}, Bull. Acad. Polon. Sci. Sér. Math. Astronom. Phys. 12 (1964), 437-438.
13. N. Noble and M. Ulmer, *Factoring functions on Cartesian products*, Trans. Amer. Math. Soc. 163 (1972), 329-340.
14. Saharon Shelah, *Remarks on cardinal invariants in topology*, General Topology and its Applications 7 (1977), 251-259.

Wesleyan University

Middletown, Connecticut 06457

---

[1]The author gratefully acknowledges support received from the National Science Foundation under grant NSF-MCS78-00912.