ARBITRARY POWERS OF THE ROOTS OF UNITY ARE MINIMAL HAUSDORFF TOPOLOGICAL GROUPS

by

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It is well-known [B, D] that minimality in the category of Hausdorff groups may fail to be preserved even in finite products. However, it has also been shown [D, G] that the group $U$ of complex roots of units is well-behaved in this respect, all finite powers of $U$ being minimal.

R. M. Stephenson asked [St2, Question 10] whether the countably infinite power of $U$ is a minimal group. We answer this question in the affirmative, as a corollary of a stronger result.

Let us recall from [H] that a topological group $G$ is called a $B(A)$ (resp., $B_r(A)$) group if every continuous, almost open (resp., and one-to-one) homomorphism from $G$ onto a Hausdorff group is open. The author, generalizing earlier results of L. J. Sulley [Su], showed [G, Theorem 1.4] that $G$ is a $B(A)$ group (resp., $B_r(A)$ group) if and only if its completion $H$ with respect to the two-sided uniformity is a $B(A)$ (resp., $B_r(A)$) group and $G \cap N$ is dense in $N$ for every closed normal subgroup $N$ of $H$ (resp., $G \cap N$ is non-trivial for every non-trivial closed normal subgroup of $H$). Stephenson showed [St1] that a condition identical to the one stated for the $B_r(A)$ property guarantees the inheritance of minimality by precompact groups from their compact completions. It therefore follows that, for precompact groups, the $B_r(A)$ property is equivalent
to minimality, and both are consequences of the \( B(A) \) property. It is now evident that the following theorem implies an affirmative answer to Stephenson's question.

**Theorem.** Let \( A \) be any set. Then \( U^A \) is a \( B(A) \) topological group.

Let \( T \) denote the circle group, \( e \) the function \( \mathbb{R} \to T \) given by \( e(t) = e^{2\pi \text{it}} \), \( (t) \) the fractional part of \( t \). A subset \( \{t_1, \ldots, t_n\} \) of \( \mathbb{R} \) is said to be rationally independent if \( r_1t_1 + \cdots + r_nt_n = 0 \) with each \( r_i \) rational implies \( r_i = 0 \) for each \( i \). We will require the following lemma which is a trivial extension of Theorem 443 of [HW].

**Lemma.** Let \( \{1, \beta_1, \ldots, \beta_n\} \) be rationally independent, \( p \) a positive integer, \( r \in \{0, 1, \ldots, p-1\} \). Then \( \{(k\beta_1), \ldots, (k\beta_n)\}: k \equiv r \ (\text{mod} \ p) \) is dense in the \( n \)-dimensional unit cube, and its image under \( e^n \) therefore dense in \( T^n \).

We now proceed to the proof of the theorem.

**Proof.** The case where \( A \) is finite has already been proved as Example 2 of [G]. We therefore assume \( A \) to be infinite. By a result of Soundararajan [So], it is sufficient to show that \( U^A \) intersects the closure of every singly-generated subgroup of \( T^A \) in a dense subgroup of that closure.

Let \( x = (x_\alpha)_{\alpha \in A} \in T^A \), \( \langle x \rangle \) the subgroup it generates, and \( X \) the closure of this subgroup in \( T^A \). If \( \text{supp} \ x = \{\alpha \in A: x_\alpha \neq 1\} \) is finite, then a similar argument to that in [G] will establish that \( U^A \cap X \) is dense in \( X \). One may then further assume, without loss of generality, that \( \text{supp} \ x = A \).
Let \( A_0 = \{ a \in A : x_a \in U \} \), \( A_1 = A \setminus A_0 \). Let \( x_a = e(\beta_a) \); then \( \beta_a \in Q \) iff \( a \in A_0 \). Let \( B \) be a maximal subset of 
\( \{ \beta_a : a \in A_1 \} \) such that \( B \cup \{ 1 \} \) is rationally independent,
\( A_2 = \{ a \in A_1 : \beta_a \in B \} \). Then, for each \( \gamma \in A_1 \setminus A_2 \), there
exist an integer \( n(\gamma) \), a finite subset \( F_\gamma = \{ \beta_1, \cdots, \beta_{\gamma n(\gamma)} \} \)
of \( B \) of minimal size, and a finite set of non-zero rational
numbers \( Q_\gamma = \{ c_1, \cdots, c_{\gamma n(\gamma)} \} \) such that \( \beta_\gamma = \sum_{i=1}^{\gamma n(\gamma)} c_i \beta_i \).

The elements of \( \langle x \rangle \) can then be characterized as follows:
\( y = (y_a) \in \langle x \rangle \) if and only if, for some integer \( r \), \( y_a = e(\beta_a r) \)
for each \( a \). In particular, for \( a \in A_1 \setminus A_2 \), \( y_a = \sum_{i=1}^{\gamma n(\gamma)} c_i \beta_i \).

Let \( K = \langle x_a \rangle_{a \in A_0} \cup A_0 \). For \( (y_a) \in \mathbb{T}^A \), let \( y_a = e(t_a) \). For \( \gamma \in A_1 \setminus A_2 \), define a subgroup \( L_\gamma \) of \( U^A \) by
\( L_\gamma = \{ (y_a) : t_\gamma = \sum_{i=1}^{\gamma n(\gamma)} c_i t_i \} \),
the coefficients being those from \( Q_\gamma \). Let \( L \) denote the inter­
section of the subgroups \( L_\gamma \) for all \( \gamma \in A_1 \setminus A_2 \).

Clearly, \( \langle x \rangle \) is a subset of \( K \times L \), and we claim that it is
in fact a dense subset. Since \( (K \times L) \cap U^A \) is dense in \( K \times L \) and
so in its closure, it would then follow that
\( \text{C}(U^A \cap \text{C}(x)) = \text{C}(U^A \cap \text{C}(K \times L)) = \text{C}(K \times L) = \text{C}(x) \).

To establish this density property, we let \( y = (y_a) \in K \times L \), \( y_a = e(t_a) \) for each \( a \). Let \( V = \prod_{a \in A} V_a \) be a neighbour­
hood of \( y \), \( E = \{ a \in A : V_a \neq T \} \), \( E_i = E \cap A_i \) for \( i = 0,1,2 \).
For \( a \in E \), let \( V'_a \) be a neighbourhood of \( t_a \) such that \( e(V'_a) \subseteq V_a \). Now, for \( a \in E_1 \setminus E_2 \), \( t_a = \sum_{i=1}^{\gamma n(\gamma)} c_i t_i \), \( c_i = m_i / n_i \),
g.c.d. \( (m_i, n_i) = 1 \). For each \( \gamma \in E_2 \), let \( G_\gamma = \{ a : \beta_\gamma = \beta_a,i(\gamma) \) for some \( i(\gamma) \in \{ 1, \cdots, n(\gamma) \} \}, \) and \( n_\gamma = \text{l.c.m.}(n_a,i(\gamma)) : a \in G_\gamma \). For each \( \gamma \in E_2 \) and \( a \in G_\gamma \), let \( d_a,i(\gamma) \) be the inte­
ger \( c_a,i(\gamma) n_\gamma \). Furthermore, let \( y_a = x_a^r \) for each \( a \in A_0 \), and
let \( p \) denote the least common multiple of the orders of the elements \( x_{\alpha} \), \( \alpha \in E_0 \). Then, for any integer \( j \), \( y_{\alpha} = x_{\alpha}^{r} = x_{\alpha}^{\cdot r+pj} \), for each \( \alpha \in E_0 \).

For each \( \alpha \in E_1 \setminus E_2 \), let \( C_{\alpha} = \{ \gamma \in A_2 : \alpha \in C_\gamma \} \), and \( C = \bigcup \{ C_{\alpha} : \alpha \in E_1 \setminus E_2 \} \). For each \( \alpha \in C \), define \( \beta_{\alpha}' = \beta_{\alpha}/n_{\alpha} \), and \( t_{\alpha}' = t_{\alpha}/n_{\alpha} \); for \( \alpha \in A \setminus C \), \( \beta_{\alpha}' = \beta_{\alpha} \) and \( t_{\alpha}' = t_{\alpha} \). It is then trivial to see that \( \{ \beta_{\alpha}' : \alpha \in A \} \cup \{ 1 \} \) is rationally independent, and that

\[
t_{\alpha} = \sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \cdot t_{\alpha}'
\]

for each \( \alpha \in E_1 \setminus E_2 \).

For all \( \alpha \in C \), select neighbourhoods \( V_{\alpha}' \) of \( t_{\alpha}' \) such that

\[
\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \cdot V_{\alpha}' \subseteq V_{\alpha}'.
\]

Let \( Y = \prod_{\alpha \in A} Y_{\alpha} \) be the neighbourhood of \((t_{\alpha}')\) given by

\[
Y_{\alpha} = \begin{cases} V_{\alpha}', & \alpha \in C \\ V_{\alpha}', & \alpha \in A \setminus C. \end{cases}
\]

Now, by the Lemma, there exists \( j \equiv r \pmod{p} \) such that

\[
e(j_{\beta_{\alpha}'}) \in e(Y_{\alpha}') \quad \text{for each} \quad \alpha \in E_1 \cup C.
\]

Then

\[
e(j_{\sum_{i=1}^{n(\alpha)} c_{ai} \beta_{ai}}) = e(j_{\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \beta_{\gamma}'})
= e(\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \beta_{\gamma}') = e(\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \cdot (y_{\gamma} + k_{\gamma})),
\]

for some integers \( k_{\gamma} \) and \( y_{\gamma} \in Y_{\gamma}' \),

\[
e(\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \cdot y_{\gamma}) \cdot 1 \subseteq e(\sum_{\gamma \in C_{\alpha}} \frac{d_{\alpha}}{n_{\alpha}} \cdot i(\gamma) \cdot y_{\gamma}) \subseteq e(V_{\alpha}') \subseteq V_{\alpha}'.
\]

It therefore follows that \( x^r \in V \), and our claim is established.

**Corollary.** \( U^A \) is a minimal Hausdorff topological group for any set \( A \).

**Remark.** Prodanov [P] has defined a topological group to be *totally minimal* if all its Hausdorff quotient groups are minimal. Since the \( B(A) \) property and precompactness are both
divisible, and together imply minimality, it follows that $U^A$ has this stronger property, as well.

**Question.** If every finite power of a group is minimal or a B(A) group, must arbitrary powers of the group have the same property?

**Bibliography**


Editor's Note:

When this paper was first received, the referee B. Banaschewski, found a major error in the proof. This fact was communicated to the author and some weeks later, Prof. Banaschewski sent to the editor a correct proof by entirely different reasoning. The existence--but not the essence--of this proof was communicated to the author who responded some months later with his own corrected version which is here printed.