SHAPE THEORY AND GEOMETRIC PROBLEMS

by

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1. Introduction

In 1968, K. Borsuk [1] introduced a new area of topology called shape theory. This theory gave a classification of compact metric spaces which was coarser than that of homotopy theory but which coincided with it on ANR's and CW complexes. The theory can be viewed as a Čech homotopy theory since its relationship to homotopy theory is analogous to the relationship between Čech homology and singular homology. Shape theory can be applied effectively to very general spaces in contradistinction to homotopy theory which requires spaces with nice local properties such as CW complexes or ANR's in order to capture their global structure.

In 1971, S. Mardešić and J. Segal [21] described an alternate approach to shape theory for compact Hausdorff spaces based on ANR-systems. If $X = (X_\alpha, p_\alpha, A)$ and $Y = (Y_\beta, q_\beta, B)$ are inverse systems of compact ANR's over cofinite directed sets with inverse limits $X$ and $Y$, respectively, then a shape map from $X$ to $Y$ is given by a homotopy class of maps of systems $(f_\beta, \phi): X \to Y$, i.e., where $\phi: B \to A$ is an increasing function and $f_\beta: Y_\phi(\beta) \to Y_\beta$ satisfies

$$f_\beta \phi(\beta') = q_\beta' f_\beta',$$

whenever $\beta \leq \beta'$. Two such maps of systems $(f_\beta, \phi)$ and $(g_\beta, \psi): X \to Y$ are homotopic if for each $\beta$ there is a $\lambda \geq \phi(\beta)$, $\psi(\beta)$ such that

$$f_\beta \phi(\beta) = g_\beta \psi(\beta).$$
K. Morita [25] generalized this approach to arbitrary topological spaces. The description which follows is the one given by Dydak and Segal in [8]. One associates with each pointed topological space \((X,x)\) a single inverse system, called the Čech system of \((X,x)\) which is denoted by \(\check{\mathcal{C}}(X,x)\). The construction of this proceeds as follows. Let \((U_a)_{a \in A}\) be the set of all locally finite numerable open coverings of \(X\) such that each \(U_a\) has exactly one member containing \(x\). If \(U_a\) is a refinement of \(U_a'\) and \(a' \neq a\), then we put \(a < a'\). For each \(a \in A\) let \((K_a,k_a)\) be the nerve of \(U_a\) (with the weak topology), where \(k_a\) is the vertex corresponding to the unique element of \(U_a\) containing \(x\). Then one can obtain a map \((X,x) \to (K_a,k_a)\) with homotopy class \(p_a: (X,x) \to (K_a,k_a)\) and a simplicial map \((K_\beta,k_\beta) \to (K_\alpha,k_\alpha)\) for \(\beta > \alpha\) with unique homotopy class \(p_\beta\) such that \(p_\beta \cdot p_\alpha = p_\alpha\) and \(p_\alpha \cdot p_\beta = p_\alpha\) for \(\alpha < \beta < \gamma\). Then the Čech system of \((X,x)\) is \(\check{\mathcal{C}}(X,x) = ((K_a,k_a), (p_\alpha, \lambda))\). This is the approach by which Čech dealt with homology.

Let \(q = (q_\beta)_{\beta \in B}: (X,x) \to ((Y_\beta,y_\beta), (\gamma_\beta', \delta))\) be a morphism of pro-HT (i.e., the pro-category of the homotopy category of pointed topological spaces). Then \(q\) is said to satisfy the continuity condition, provided: (1) (factoring) for any map \(f: (X,x) \to (K,k)\) into \((K,k)\) which is homotopy equivalent to a pointed CW complex there is a map \(q_\beta: (Y_\beta,y_\beta) \to (K,k)\) with \([q_\beta] \cdot q_\beta = [f]\) and (2) (short tails) if \(g,h: (Y_\beta,y_\beta) \to (K,k)\) are two maps such that \([g] \cdot q_\beta = [h] \cdot q_\beta\), then for some \(\beta' \geq \beta\), \([g] \cdot q_\beta' = [h] \cdot q_\beta'\). Then the main
property of Čech systems is that \( \varphi_X = (\varphi_a)_{a \in A} : (X,x) \to \check{C}(X,x) \) satisfies the continuity condition.

Two important shape invariants are Čech homology and cohomology [21]. It is also possible to describe new continuous functors for an arbitrary topological space \( X \) such as the shape groups by taking inverse limits of inverse systems of homotopy groups of inverse systems associated with \( X \). Further, one can use the systems themselves without passing to the limit to obtain the homotopy pro-groups. These homotopy pro-groups are an important shape invariant. Borsuk [3] also introduced a shape invariant called movability. This is a generalization of the notion of ANR. Mardesić and Segal [23] described movability in terms of ANR-systems. An inverse system \( \check{X} = (X_a, p_a^a, A) \) is said to be movable provided for any \( \alpha \in A \) there is an \( \alpha' \geq \alpha \) such that for each \( \alpha'' \geq \alpha' \) there is a morphism \( r : X_{\alpha'} \to X_{\alpha''} \) with \( p_{\alpha'}^\alpha r = p_{\alpha''}^\alpha \). This definition applies in any pro-category.

A space \( (X,x) \) is called movable provided \( \check{C}(X,x) \) is movable.

The importance of movability stems from the fact that in its presence, one may take the inverse limit of a system without losing algebraic information about the system.

To illustrate this consider the following shape version of the classical Whitehead theorem. (This version is due to M. Moszyńska [27] and was improved upon by Mardesić [20] and J. Keesling [15].

**Whitehead Theorem.** Let \( f : (X,x) \to (Y,y) \) be a shape map of pointed continua. If \( f \) induces isomorphism \( \text{pro}-\pi_k(f) : \text{pro}-\pi_k(X,x) \to \text{pro}-\pi_k(Y,y) \) of the homotopy pro-groups for
k \leq m + 1 \text{ and } \max(\text{ddim } X, \text{ddim } Y) \leq m, \text{ then } f \text{ is a shape equivalence.}

Note that the shape version applies to continua while the homotopy version only applied to spaces with strong local properties such as CW complexes and that the homotopy pro-groups have replaced the homotopy groups.

Recall that the deformation dimension, \( \text{ddim } X \), of a space \( X \) is the minimum \( n \) such that any map of \( X \) into a CW complex \( K \) is homotopic to one whose image lies in the \( n \)-skeleton of \( K \).

The following is a movable version of the Whitehead Theorem.

**Corollary.** Let \( f: (X,x) \rightarrow (Y,y) \) be a shape map of pointed continua such that \( \pi_k(f): \pi_k(X,x) \rightarrow \pi_k(Y,y) \) is an isomorphism for \( k \leq m + 1 \). If \( (X,x) \) and \( (Y,y) \) are movable and \( \max(\text{ddim } X, \text{ddim } Y) \leq m \), then \( f \) is a shape equivalence.

2. **Locally Well-Behaved Shape Representatives**

In this section we discuss the question of when a continuum has the shape of (1) a locally connected continuum or (2) a CW complex. Borsuk [2] introduced an \( n \)-dimensional stratification of movability called \( n \)-movability. The pointed \( 1 \)-movable case is of special interest because a continuum possessing this property has the shape of some locally connected continuum and every locally connected continuum is pointed \( 1 \)-movable. Moreover, such continua can be characterized by a purely algebraic property in terms of their first homotopy pro-group.
An inverse system \( \{(X_\alpha, x_\alpha), p_\alpha^\alpha, \mathcal{A}\} \) of the homotopy category of pointed topological spaces and maps preserving base points is n-movable iff for each \( \alpha \) there is an \( \alpha' \geq \alpha \) such that for all \( \alpha'' > \alpha \), and any homotopy class \( f: (K, k) \to (X_{\alpha''}, x_{\alpha''}) \) where \( K \) is an n-dimensional CW complex, there exists a homotopy class \( g: (K, k) \to (X_{\alpha'}, x_{\alpha'}) \) with
\[
p_{\alpha''}^\alpha \cdot g = p_{\alpha'}^\alpha \cdot f.
\]

Then a pointed topological space \((X, x)\) is n-movable if its Čech system \( \check{C}(X, x) \) is n-movable. Finally, a continuum \( X \) is called pointed l-movable provided \((X, x)\) is l-movable for each \( x \) in \( X \).

A pro-group \( G = \{G_{\alpha}, p_\alpha^\alpha, \mathcal{A}\} \) is said to satisfy the Mittag-Leffler condition provided for any \( \alpha \) in \( \mathcal{A} \), there exists an \( \alpha' \geq \alpha \) such that for any \( \alpha'' \geq \alpha' \) we have
\[
p_{\alpha''}^\alpha (G_{\alpha''}) = p_{\alpha'}^\alpha (G_{\alpha'}).\]

Further, a pro-group \( G \) is said to be stable if it is isomorphic, as a pro-group, to a group.

**Lemma.** A pointed continuum \((X, x)\) is l-movable iff \( \text{pro-}_1(X, x) \) satisfies the Mittag-Leffler condition.

**Theorem 1.** [7] Let \((X, x)\) and \((Y, y)\) be pointed continua. If \( \text{Sh}(X) = \text{Sh}(Y) \) and \((X, x)\) is l-movable, then \( \text{Sh}(X, x) = \text{Sh}(Y, y) \).

**Corollary.** A continuum \( X \) is pointed l-movable iff \((X, x)\) is l-movable for some \( x \) in \( X \).

Note that in general \( \text{Sh}(X) = \text{Sh}(Y) \) (i.e., \( X \) and \( Y \) have
the same unpointed shape) does not imply that $\text{Sh}(X,x) = \text{Sh}(Y,y)$ (i.e., $X$ and $Y$ have the same pointed shape) for all $x$ in $X$ and $y$ in $Y$. However, Dydak [7] has shown that pointed 1-movability is an invariant of unpointed shape.

The following result due to Krasinkiewicz [18] characterizes those continua which have the shape of a locally connected continuum.

**Theorem 2.** For a continuum $X$ the following conditions are equivalent:

(a) there exists a decreasing sequence $X_1, X_2, X_3, \ldots$ of locally connected continua such that $X = \bigcap_{j=1}^{\infty} X_j$ and $X_{j+1}$ is a strong deformation retract of $X_j$ for $j \geq 1$,

(b) $X$ has the shape of a locally connected continuum,

(c) $X$ is pointed 1-movable.

By the Lemma and Theorem 1, condition (c) is equivalent to pro-$\pi_1(X,x)$ satisfying the Mittag-Leffler condition. This result provides a direct computational means to determine which continua have the shape of a locally connected continuum. Among the known classes of pointed 1-movable continua are (1) arcwise connected continua [19], (2) hereditarily decomposable continua [17], (3) subcontinua of 2-manifolds [17], [24], (4) continuous images of pointed 1-movable continua [17], [24]. On the other hand, the dyadic solenoid is not pointed 1-movable.

S. Ferry [13] recently generalized Krasinkiewicz's result to homotopy locally $n$-connected continua as follows.

**Theorem 3.** A continuum $X$ has the shape of an $\text{LC}\,^n$ con-
tinuum iff pro-$\pi_k(X)$ is stable for $0 \leq k \leq n$ and Mittag-Leffler for $k = n + 1$.

Borsuk [3] also introduced a shape version of ANR's which is more restrictive than movability and which is called FANR's. We refer to these spaces as absolute neighborhood shape retracts (ANSR's). An important property of pointed ANSR's is that they have the shape of CW complexes. This allows the use of the theory of CW complexes to investigate their geometry. D. A. Edwards and R. Geoghegan [10] have shown that a pointed connected space $(X, x)$ of finite shape dimension has the shape of a pointed CW complex iff each of its pro-groups pro-$\pi_n(X, x)$ is stable. However, it is not known whether ANSR's are pointed ANSR's. This question has received considerable attention. Geoghegan [14] has shown that a connected ANSR $X$ is a pointed ANSR iff the first derived limit $\lim^1\text{pro-}\pi_1(X, x) = 0$. J. Dydak and P. Minc [7] have described some examples which come close to answering the question.

3. Shape and Some Geometric Problems

One way in which shape theory is related to geometric problems is through the consideration of images of ANR's under CE (or cell-like) mappings (a mapping $f: X \to Y$ of $X$ onto $Y$ is said to be a CE map if the inverse image of each point has trivial shape). Between ANR's and, in particular, manifolds CE maps worked well and play a central role in the work of L. C. Siebenmann [28], R. D. Edwards, and J. E. West [32]. However, J. L. Taylor's example [29] of a CE map from the Kahn space (an infinite dimensional acyclic
space of nontrivial shape) onto the Hilbert cube which is not a shape equivalence showed the need to limit CE maps in some way in the more general setting. To do this G. Kozlowski [16] introduced the notion of hereditary shape equivalence. A mapping $f: X \to Y$ of $X$ onto $Y$ is a hereditary shape equivalence iff $f|f^{-1}(C): f^{-1}(C) \to C$ is a shape equivalence for all closed subsets $C$ of $X$. By restricting $C$ to the points of $X$ one sees that hereditary shape equivalences are CE maps. Hereditary shape equivalences behave well with respect to quotients and agree with CE maps on spaces which have a strong local structure (e.g., CE maps are hereditary shape equivalences when they map between ANR's or when the range has finite dimension). Kozlowski used this shape theoretic notion to give the following elegant characterization of the CE images of ANR's.

**Theorem 5.** If $f: X \to Y$ is a CE map and $X$ is an ANR, then $Y$ is an ANR iff $f$ is a hereditary shape equivalence.

The behavior of CE maps with respect to dimension has been of considerable interest. Formally the question can be asked as follows.

**Question 1.** Does there exist a CE map which raises dimension? In [31] J. J. Walsh has pointed out how Question 1 is related to the following classical problem.

**Question 2.** Does there exist an infinite dimensional compactum with finite cohomological dimension?

If such a compactum existed it could not contain finite
dimensional subsets with dimension greater than the cohomological dimension of the compactum. Walsh [30] has recently constructed infinite dimensional compacta which do not contain n-dimensional \((n \geq 1)\) subsets and has given [31] a method to show many of the known examples have infinite cohomological dimension. The \textit{cohomological dimension} of a space \(X\), \(\text{cdim} X\), is defined to be the minimum integer \(n\) such that, whenever \(m \geq n\) and \(A\) closed in \(X\), the homomorphism \(i^*: H^m(X;G) \to H^m(A;G)\) induced by the inclusion \(i: A \subset X\) is onto. Moreover, it follows from the fact that cohomological dimension and dimension agree on finite dimensional spaces and the Vietoris-Begle mapping theorem that the image of a CE dimension raising map would be infinite dimensional and would have finite cohomological dimension.

In [12] R. D. Edwards indicates why Questions 1 and 2 are equivalent. Moreover, he outlines a program to find a counterexample in terms of an infinite collection of mappings between spheres whose finite compositions are all essential. Finally, Kozlowski has shown that the problem of CE dimension raising maps can be formulated in shape theory as follows.

\textit{Question 3.} If \(X\) is a finite dimensional compactum and \(f: X \to Y\) is a CE map, then is \(f\) a shape equivalence?

4. \textbf{Strong Shape and Some Geometric Problems}

Another category, called the strong shape category, is also closely related to geometric problems. There has been considerable interest in the strong shape category because
it behaves better with respect to pairs and extensions that the shape category. Another reason for this interest is that if every shape equivalence is a strong shape equivalence, then every ANSR is a pointed ANSR. The exact relationship between the equivalence of these categories is unresolved at the moment (see [9]). R. Geoghegan [14] has given a detailed account of the pointed versus the unpointed case and its importance is shape theory. D. A. Edwards and H. M. Hastings [11] have given a categorical treatment of strong shape theory. The objects in their category are compact $Z$-sets in the Hilbert cube $Q$ and the morphism between such compacta $X$ and $Y$ are proper homotopy classes of maps from $Q - X$ to $Q - Y$. Most likely they were influenced in this approach by T. A. Chapman's [5] description of the shape category as one whose objects are complements of compact $Z$-sets in $Q$ and whose morphisms are weak proper homotopy classes of proper maps. A. Calder and H. M. Hastings [4] have recently announced a purely categorical description of the strong shape category.

Dydak and Segal in [9] describe the strong shape category as one whose objects are compacta $X$ and whose morphisms (strong shape maps) from $X$ to $Y$ are natural transformations from $\Pi_{\text{CTel}Y}$ to $\Pi_{\text{CTel}X}$ where $\Pi_{\text{CTel}(X,A)}: W_P \to E_n$ is the functor sending a pair of ANR's $(P,R)$ to the set of morphisms in the proper homotopy category from the contractible telescope $\text{CTel}(X,A)$ to $(P,R)$. The category obtained is isomorphic to the one previously described by Edwards and Hastings. However, Dydak and Segal concentrate on finding geometric conditions for a map to induce a strong shape
equivalence. They use proper homotopy theory to obtain geometric information about the strong shape category. For example, they are able to reduce the question of whether a morphism is a strong shape equivalence to verifying whether other maps are shape equivalences. This takes various forms, two of which follow.

**Theorem 6.** Let $f: X \rightarrow Y$ be a map of compacta. Then $f$ induces a strong shape equivalence iff for each compactum $Z$ containing $X$ the natural projection $p: Z \rightarrow Z \cup_f Y$ induces a shape equivalence.

**Theorem 7.** Let $f: X \rightarrow Y$ be a map of compacta. Then $f$ induces a strong shape equivalence iff $f: X \rightarrow Y$ and $\hat{f}: \hat{M}(f) \rightarrow Y$ induce shape equivalences.

In the above $\hat{M}(f)$ denotes the double mapping cylinder of $f: X \rightarrow Y$ which is defined as the adjunction space

$$\hat{M}(f) = (X \times [-1,1]) \cup_\phi (Y \times \{-1,1\})$$

where

$$\phi = f \times 1: X \times \{-1,1\} \rightarrow Y \times \{-1,1\}.$$ 

The image of $(u,t)$ under the quotient map $q: (X \times [-1,1]) \cup (Y \times \{-1,1\}) \rightarrow \hat{M}(f)$ is denoted by $[u,t]$. The map $\hat{f}: \hat{M}(f) \rightarrow Y$ is defined by $\hat{f}[u,t] = f(u)$, if $(u,t) \in X \times [-1,1]$ and by $\hat{f}[u,t] = u$, if $(u,t) \in Y \times \{-1,1\}$. Theorem 7 is obtained from Theorem 6 using a technique due to G. Kozlowski [16].

Dydak and Segal first prove a strong shape version of the Fox theorem. Then the question of whether a strong shape morphism is a strong shape equivalence is reduced to the consideration of simpler cases, namely, what happens
with maps induced by inclusions. As an application of their results they are able to determine when various classes of mappings induce strong shape equivalences.

**Theorem 8.** A hereditary shape equivalence of compacta induces a strong shape equivalence.

This theorem follows from Theorem 6 since Kozlowski has shown that any extension of a hereditary shape equivalence is a shape equivalence.

**Theorem 9.** If \( f: X \to Y \) is a CE map and \( \text{ddim } X, \text{ddim } Y < \infty \), then \( f \) induces a strong shape equivalence.

**References**


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