
TOPOLOGY PROCEEDINGS



Volume 4, 1979

Pages 385–392

<http://topology.auburn.edu/tp/>

CONCERNING EQUIVALENCIES OF ALMOST CONTINUOUS AND CONNECTIVITY FUNCTIONS OF BAIRE CLASS 1 ON PEANO CONTINUA

by

B. D. GARRETT AND D. L. ALEXANDER

Topology Proceedings

Web: <http://topology.auburn.edu/tp/>

Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA

E-mail: topolog@auburn.edu

ISSN: 0146-4124

COPYRIGHT © by Topology Proceedings. All rights reserved.

**CONCERNING EQUIVALENCIES OF ALMOST
CONTINUOUS AND CONNECTIVITY FUNCTIONS
OF BAIRE CLASS 1 ON PEANO CONTINUA**

B. D. Garrett and D. L. Alexander

Kuratowski and Sierpinski showed, in 1922 [6], that whenever a function of Baire class 1, $f: I \rightarrow I$, has the Darboux property, then that function is a connected subset of the plane. For real functions with domain a connected subset of the real line, the property of a function being a connected subset of the plane, and the property of a function being a connectivity function are the same. Stating the Kuratowski-Sierpinski result in this terminology we have: a necessary and sufficient condition that the Baire class 1 function $f: I \rightarrow I$, be a Darboux function is that it be a connectivity function. Intuitively, when considering various generalizations of continuity, there are three classes of discontinuous functions which seem to be inherently comparable or interrelated: Darboux functions, connectivity functions, and almost continuous functions. Thus there was the obvious question of whether or not the Kuratowski-Sierpinski characterization could be extended to include the almost continuous functions. Brown has answered that question affirmatively [1]. Herein we are concerned with the extension of such properties to cases where the domain space of the functions is other than a subset of the real line.

We will consider only real functions having domain a Peano Continuum (a compact separable locally connected metric

space). For the definitions or properties of such things as "hereditarily locally connected space," "continuum of convergence," etc., see [10]. Most of the notation and terminology, such as R being the real line or I being the closed unit interval, is standard or it will be clear from context. When each of X and Y is a topological space and M is a subset of the product space $X \times Y$, then $(M)_X$ denotes the projection of M into X .

Definition 1. The function $f: X \rightarrow Y$ is said to be a *Darboux function* or to have the *Darboux property* if, whenever C is a connected subset of X , then $f(C)$ is connected.

Definition 2. The function $f: X \rightarrow Y$ is said to be a *connectivity function* provided that, whenever C is a connected subset of X , then $f|_C$ is connected.

Definition 3. The function $f: X \rightarrow Y$ is said to be an *almost continuous function* provided that, whenever 0 is an open set in $X \times Y$ which contains f , then 0 contains a continuous function from X to Y .

There are examples in the literature which can be used to show that, for functions $f: X \rightarrow Y$, where X is a Euclidean space other than the real line, equivalences like those found by Kuratowski and Sierpinski or by Brown do not hold [1] [5]. Using the Theorem below, Example 1 of [5] can be shown to be a Baire class 1, almost continuous, Darboux function which is not a connectivity function. There is some character to the real line not possessed by the other Euclidean spaces which allows those equivalences, or, looking at it from the

other side, such equivalences impose restrictions on the domain of the functions which rule out the other Euclidean spaces. Clearly though, there are other spaces (such as a simple triod) where those same equivalences hold.

It is interesting to consider problems of this type. Generally, if the Baire class 1 function $f: X \rightarrow Y$ belongs to some one of the classes determined by Definition 1, Definition 2, or Definition 3 only in case it also belongs to some other of those classes, then what sort of space is X ?

After building (or in other cases being unable to build) a number of examples, a conjecture resulted: for connectivity and almost continuity to be equivalent for real functions of Baire class 1, having domain a topological space X , it is necessary and sufficient that X be hereditarily locally connected. The question of the sufficiency of this conjecture remains open; however, we show that the necessity is true whenever X is a Peano Continuum.

Theorem. Suppose X is a Peano Continuum and the function $f: X \rightarrow \mathbb{R}$ is of Baire class 1. In order that it be true that f is almost continuous if and only if f is a connectivity function, then X must be hereditarily locally connected.

Proof. Under the hypotheses, assume that X is not hereditarily locally connected and so has a continuum of convergence. Then X has a nondegenerate subcontinuum M and there is a sequence N_1, N_2, N_3, \dots of subcontinua of X such that no two of them intersect and no one of them intersects M , and the sequence of continua converges to M [10]. Denote by P a point of M and by T an open set of X containing P but

not all of M . Select a subsequence M_1, M_2, M_3, \dots , of N_1, N_2, N_3, \dots , each term of which intersects T .

There is a monotonic collection (D_1, D_2, D_3, \dots) of open subsets of X such that

- (1) D_1 is X ,
- (2) for each integer $i \geq 2$, \bar{D}_i is a subset of D_{i-1} ,
- (3) M is the common part of all of them, and
- (4) for each integer $i \geq 1$, M_i is a subset of D_i but has no point in \bar{D}_{i+1} .

For each positive integer n , A_n is the set $M + \text{Bd}(D_2) + \text{Bd}(D_3) + \text{Bd}(D_4) + \dots + \text{Bd}(D_{n-1})$ and B_n is the set $M_1 + M_2 + M_3 + \dots + M_n$. Because each of these is closed, by using Urysohn's lemma, there is a continuous function $g_n: X \rightarrow \mathbb{R}$ such that g_n is 0 on A_n and 1 on B_n . It is clear that this sequence of g_n 's need not converge; however, suppose f_1, f_2, f_3, \dots , is a sequence such that: f_1 is g_1 , and, if n is a positive integer greater than 1, f_n is f_{n-1} on $X - D_n$ and is g_n on D_n . Each of these f_n 's will be continuous on all of X and whenever x is in X , there is an integer N such that, if n and m are greater than N , $f_n(x) = f_m(x)$. Denote by f the function which is the limit of the sequence of f_n 's so that we have a function $f: X \rightarrow \mathbb{R}$ which is of Baire class 1.

We will show that this Baire class 1 function f is almost continuous, which, by hypothesis, requires it to be a connectivity function and then reach a contradiction by finding a connected subset C of X such that $f|_C$ is not connected.

To show that f is almost continuous, consider X and its various subsets as objects in $X \times \mathbb{R}$, i.e., M is $M \times 0$, D_1 is $D_1 \times 0$, and so on. Thus, if U is an open set in $X \times \mathbb{R}$

containing $M \times 0$ (that is, U contains M), then $U \cdot (X \times 0)$ is open in D_n plus its boundary relative to X . Suppose V is an open subset of $X \times R$ containing f , and suppose the subset U of V is open in $X \times R$ and contains $M \times 0$. There is a positive integer k such that k is in $U \cdot (X \times 0)$.

Define $g: X \rightarrow R$ as follows: $g(x) = 0$ if x is in \bar{D}_k and $g(x) = f_k(x)$ for x not in \bar{D}_k . Notice that since f_k is 0 on $\bar{D}_k - D_k$, then this function is continuous. Clearly each point $(x, g(x))$, with x in \bar{D}_k is in U and therefore in V , by the construction of the f_i 's, outside of \bar{D}_k , g is f so that this function $g: X \rightarrow R$ is continuous and is a subset of V . Thus f is almost continuous and must be a connectivity function.

We can show that f cannot be a connectivity function if we can find a connected subset C of X such that $f|_C$ is not connected. Describe such a connected set as follows. Suppose A is an arc containing the point P of M , intersecting each M_i , and which is a subset of the open set T of X containing P but not all of M . To the set $A + M_1 + M_2 + M_3 + \dots$ add a single point q of M which is not in \bar{T} and call the resulting connected set C . It is easy to see that each sequence of points of f which converges in $X \times R$ to a point (q, y) is not the point $(q, f(q))$. The point $(q, f(q))$ is an isolated point of $f|_C$. Thus $f|_C$ cannot be connected and so f is not a connectivity function even though it is almost continuous. With this contradiction we have proved that X must be hereditarily locally connected.

In order to more concisely state some problems concerning

the possibilities of extending the Kuratowski-Sierpinski and the Brown results, we use the following definitions.

Definition 4. Suppose X is a topological space.

(1) If for each Baire class 1 function $f: X \rightarrow \mathbb{R}$, f is a connectivity function if and only if f is a Darboux function, then X is said to be a *KS-space* or to have *property KS*.

(2) If for each Baire class 1 function $f: X \rightarrow \mathbb{R}$, f is a Darboux function if and only if f is almost continuous, then X is said to be a *B-space* or to have *property B*.

(3) If for each Baire class 1 function $f: X \rightarrow \mathbb{R}$, f is a connectivity function if and only if it is almost continuous, X is said to be a *G-space* or to have *property G*.

Problems:

(1) We know that I is a KS-space, a B-space and a G-space, and we have shown above that other n -cells are not G-spaces. From known examples it seems to be true that if $n > 1$, then I^n would not be either a B-space or a KS-space. What sort of spaces will be those defined in definition 4?

(2) Does hereditary local connectedness imply property G?

(3) Are the properties defined in definition 4 equivalent?

(4) Stallings [9] showed that $f: I^n \rightarrow \mathbb{R}$ is almost continuous if it is a connectivity function, for $n > 1$, and f is a connectivity function if it is an almost continuous function, for $n = 1$. Would it be true in general that, in a G-space almost continuity implies connectivity, and in a

non-G-space connectivity implies almost continuity? (What about B-spaces and KS-spaces?)

(5) Extending local characterizations [3] [4] of the Darboux property and connectivity to n -cells other than I has been successful only with severe restrictions [2] [5], apparently because I is the only one of them which is hereditarily locally connected. It seems reasonable to conjecture that for G-spaces, and probably KS-spaces and B-spaces, extensions of local properties such as are found in [3], [4], [7], and [8] could be proved.

References

1. J. B. Brown, *Almost continuous Darboux functions and Reed's pointwise convergence criteria*, Fund. Math. 86 (1974).
2. A. M. Bruckner and J. B. Bruckner, *Darboux transformations*, Trans. Amer. Math. Soc. 128 (1976).
3. A. M. Bruckner and J. Ceder, *Darboux continuity*, Jahr. Deutsch. Math. Verein. 67 (1965).
4. B. D. Garrett, D. Nelms, and K. R. Kellum, *Characterizations of connected real functions*, Jahr. Deutsch. Math. Verein. 67 (1971).
5. R. G. Gibson and K. R. Kellum, *Darboux (β) functions, connectivity (β) functions, and functions of Baire class 1*, Colloq. Math. 35 (1976).
6. C. Kuratowski and W. Sierpinski, *Les fonctions de class I et les ensembles punctiformes*, Fund. Math. 3 (1922).
7. M. H. Miller, Jr., *Characterizations of real functions by continua*, Fund. Math. 93 (1976).
8. H. Rosen, *Connectivity points and Darboux points of real functions*, Fund. Math. 89 (1975).
9. J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. 47 (1959).
10. G. T. Whyburn, *Analytic topology*, Amer. Math. Soc. Colloq. Publs. 28 (1963), Providence, R.I.

University of Alabama
Tuscaloosa, Alabama 35401
and
University of Montevallo
Montevallo, Alabama 35115