A DIFFERENTIABLE, PERFECTLY NORMAL, NONMETRIZABLE MANIFOLD

by

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In answer to a question originally raised by Alexandroff in [A], Rudin and Zenor, using the continuum hypothesis, displayed an example of a perfectly normal, hereditarily separable, non-metrizable topological manifold [R,Z]. In this paper, we show that the Rudin-Zenor manifold can be constructed so that it is analytic. A key step in our construction is a modification of a theorem of Brown [B] which is interesting in its own light; namely, we show that if a differentiable manifold M has an atlas \{(V_i, \phi_i) \mid i \in \omega_0\} such that \(V_{i+1} \supseteq V_i\) and \(\phi_i(V_i) = \mathbb{R}^n\) for all \(i \in \omega_0\), then M is diffeomorphic to \(\mathbb{R}^n\).

The construction of the manifold follows very closely that of [R,Z] and we recommend that the reader be familiar with that paper before proceeding.

Let X be a set, and let n be a fixed positive integer.

A chart is a pair \((U, \phi)\) where \(\phi: U \to \mathbb{R}^n\) is an injective function of a subset U of X onto an open subset \(\phi(U)\) of \(\mathbb{R}^n\).

Two charts \((U, \phi)\), \((V, \psi)\) are compatible, if \(\phi(U \cap V)\) and \(\psi(U \cap V)\) are open subsets of \(\mathbb{R}^n\) and \(\psi^{-1}\mid\phi(U \cap V): \phi(U \cap V) \to \psi(U \cap V)\) is a diffeomorphism.

An atlas on the set X is a collection \\{(U_j, \phi_j) \mid j \in J\}\.

*This author's research was partially supported by NSF Grant #MSC 7813270.
of charts such that \( X = \bigcup \{ U_j \mid j \in J \} \) and any two charts are compatible.

A differential structure \( \mathcal{D} \) on a set \( X \) is a maximal atlas. It is clear that any atlas is contained in a unique differential structure which is said to generate.

If \( \mathcal{A} \) is an atlas on the set \( X \), it is also clear that there is a unique topology on \( X \) with the property that \( \phi : U \to \phi U \) is a homeomorphism of the open set \( U \) onto \( \phi U \) for every chart \( (U, \phi) \).

A smooth manifold is a set \( X \) together with a differential structure \( \mathcal{D} \) or \( X \); notation: \( (X, \mathcal{D}) \). When there is no danger of confusion, one simply refers to the smooth manifold \( X \).

Let \( D(r) = \{ u \in \mathbb{R}^n \mid |u| < r \} \), and let \( M \) be a smooth \( n \)-manifold. A subset \( D \) of \( M \) is said to be an \( n \)-disk, provided there is a chart \( (U, \phi) \) of \( M \) such that \( \phi D = D(r) \) for some positive number \( r \). (This definition allows us to avoid some technicalities regarding differentiability on sets which are not open.)

If \( D \) is an \( n \)-disk in \( M \), then a map \( f : M \to M \) is said to be a radial diffeomorphism in \( D \), if there exist a chart \( (U, \phi) \) of \( M \), a positive number \( \varepsilon \), and a diffeomorphism \( \lambda : \mathbb{R} \to \mathbb{R} \) such that \( \phi D = D(1) \), \( \lambda(t) = t \) for all \( t < \varepsilon \) and all \( t > 1 - \varepsilon \), \( f(x) = x \) for all \( x \in M - D \), and \( f(x) = \phi^{-1} \lambda \phi(x) \) for \( x \in D \), where \( \Lambda : \mathbb{R}^n \to \mathbb{R}^n \) if defined by \( \Lambda(u) = \lambda(|u|)u/|u| \) if \( u \neq 0 \) and \( \Lambda(0) = 0 \). Because \( f \) is the identity on \( M - D \) and a diffeomorphism of \( \text{Int} D \), \( f : M \to M \) is in fact a diffeomorphism.

\textbf{Lemma 1.} \textit{If} \( D_1, D_2, D_3, D_4 \) \textit{are} \( n \)-disks in a smooth
manifold \( M \) such that \( D_i \subset \text{Int } D_{i+1} \) for \( i = 1,2,3 \), then there is a diffeomorphism \( f: M \to M \) such that \( f(x) = x \) for \( x \in D_1 \cup (M - D_4) \) and \( \text{Int } fD_2 \supset D_3 \).

Proof. There is a radial diffeomorphism \( g: M \to M \) in \( D_4 \) which is the identity on a nonempty open subset \( B \) of \( \text{Int } D_1 \) and which maps \( D_3 \) into \( D_1 \), and there is a radial diffeomorphism \( h: M \to M \) in \( D_2 \) which maps \( D_1 \) into \( V \). Put \( f = h^{-1}g^{-1}h \). If \( x \in D_3 \), then \( h(x) \in D_3 \) and \( gh(x) \in D_1 \) and consequently \( h^{-1}gh(x) \in \text{Int } D_2 \); hence \( f^{-1}D_3 \in \text{Int } D_2 \), and therefore \( D_3 \subset g(\text{Int } D_2) \).

Theorem 1. If a differentiable manifold \( M \) has an atlas \( \{(U_i, \phi_i) | i \in \omega_0 \} \) such that \( U_i \subset U_{i+1} \) and \( \phi_i U_i = \mathbb{R}^n \) for all \( i \in \omega_0 \), then \( M \) is diffeomorphic to \( \mathbb{R}^n \).

Proof. Let \( h_i = \phi_i^{-1}: \mathbb{R}^n \to U_i \subset M \). From the hypothesis that \( U_i \subset U_{i+1} \) for \( i \in \omega_0 \) it follows that there is a strictly increasing sequence of positive integers \( r_i, i \in \omega_0 \) such that \( U(h_iD(r_i) | i \in \omega_0) = M \) and \( h_iD(r_i) \subset \text{Int } h_{i+1}D(r_{i+1}) \) for \( i \in \omega_0 \). Put \( Q_i = h_iD(r_i) \).

We assert that there exist a sequence of diffeomorphisms \( f_i: M \to M, i \in \omega_0 \) and a strictly increasing sequence of positive numbers \( s_i, i \in \omega_0 \) with limit \( r_1 \) such that \( A(i): f_i \) is the identity on \( M - Q_i+1 \) and on \( f_{i-1} \cdots f_1 f_0 h_1 D(s_{i-1}) \) and such that \( B(i): f_i \cdots f_1 f_0 h_1 D(s_i) \supset Q_i \). To verify this assertion assume inductively that \( f_i \) and \( s_i \) for \( i = 0,1,\ldots,k \) satisfy \( A(i) \) and \( B(i) \) for \( i = 0,1,\ldots,k \). Since \( f_k \cdots f_1 f_0 Q_1 \subset \text{Int } Q_{k+1}, \) there is \( s_{k+1} > s_k \) such that \( 0 < r_i - s_{k+1} < l/(k+1) \), and the lemma applies to \( D_1 = f_k \cdots f_1 f_0 h_1 D(s_k), \ D_2 = f_k \cdots f_1 f_0 h_1 D(s_{k+1}), \ D_3 = Q_{k+1}, \) and \( D_4 = Q_{k+2} \) to provide
a diffeomorphism $f_{k+1}: M \to M$ such that $A(k+1)$ and $B(k+1)$ hold.

To complete the proof of the Theorem, define $F: \text{Int } Q_1 \to M$ by $F(x) = \lim_{k \to \infty} F_k(x)$ where $F_k = f_k \cdots f_1 f_0: M \to M$. Since $F(x) = F_k(x)$ for $x \in h_1 D(s_k)$, $F$ is well-defined and clearly a homeomorphism onto $M$. Since $F$ is a diffeomorphism on each of the open sets $\text{Int } h_1 D(s_k)$, $k \in \omega_0$, it is a diffeomorphism of $\text{Int } Q_1$ (which is diffeomorphic to $\mathbb{R}^n$) onto $M$.

Lemma 2. Any closed smooth embedding $\mathbb{R} \to \mathbb{R}^2$ extends to a diffeomorphism of $\mathbb{R}^2$ onto itself.

Proof. Any closed embedding of $\mathbb{R}$ into $\mathbb{R}^2$ extends to a closed embedding $f: \mathbb{R} \times [-2,2] \to \mathbb{R}^2$ by means of the Collaring Theorem.

Take a rectilinear triangulation $T$ of $\mathbb{R}^2 \setminus f(\mathbb{R} \times \{0\})$. The 1-simplices of $T$ which are not contained in $f(\mathbb{R} \times [-1,1])$ comprise a sequence $\{A(j) | j \in \omega\}$ with the property that for any compact set $K$ in $\mathbb{R}^2$ there is an index $j(K)$ such that $A(j) \cap K = \emptyset$ for all $j \geq j(K)$.

For each positive real number $r$ define the band $B(r) = \mathbb{R} \times [-2 + 1/r, 2 - 1/r]$. We claim there is a sequence of closed embeddings $F_n: \mathbb{R} \times [-2,2] \to \mathbb{R}^2$ ($n \in \omega$) such that $F_0 = f$ and for all $n \in \omega$:

1. $F_{n+1}(x) = F_n(x)$ for the points $x$ of $B(n)$ and
2. $F_n(B(n)) = A(j)$ for all $j < n$.

If such a sequence exists, define $F: \mathbb{R} \times (-2,2) \to \mathbb{R}^2$ by $F(x) = \lim_{n \to \infty} F_n(x)$; then $F$ extends $f|B(1)$ and is a diffeomorphism onto an open set which contains every 1-simplex.
of the triangulation $T$ of $\mathbb{R}^2 - f(\mathbb{R} \times 0)$ and hence by simple-connectivity every point of $\mathbb{R}^2$. It follows easily that there is a diffeomorphism of $\mathbb{R}^2$ onto itself extending the original closed embedding $\mathbb{R} \to \mathbb{R}^2$.

The claim is proved by induction. Assume $F_n$ has been obtained satisfying (2).

If $A(n) \cap F_n(B(n)) = \emptyset$, it is easy to construct a diffeomorphism $f$ of $\mathbb{R}^2$ onto itself so that $g$ is the identity on $F_n(B(n))$ and $g(A(n)) \subset F_n(B(n+1))$. In this case, take $F_{n+1} = g^{-1}F_n$. If $A(n) \cap F_n(B(n)) \neq \emptyset$, there is a finite sequence of closed subintervals $\{C_1, C_2, \ldots, C_r\}$ so that $A(n) - \bigcup\{C_i | i \leq r\}$ is contained in $F_n(B(n+\frac{1}{2}))$ and so that $C_i \cap F_n(B(n)) = \emptyset$ for $i \leq r$. By a preliminary diffeomorphism, if necessary, we may assume the set of endpoints of $C_i$ is a subset of $F_n(B(n+\frac{1}{2}))$ for $i \leq r$. For each $C_j$ let $C'_j$ be an arc lying in $F(B(n+\frac{1}{2})) - F(B(n))$ so that $C_j \cup C'_j$ is a simple closed curve so that $C'_j \cap C_i = \emptyset$ for all $i \neq j$. Let $M = \{i \leq r | \text{if } j \neq i, C_i \text{ is not a subset of the bounded domain of } C_j \cup C'_j\}$. For each $i \in M$, let $C''_i$ be an arc so that $C_i \cup C'_i \cup C''_i$ is a $0$-curve with $C_i$ as the cross-arc such that if $i \neq j$ are in $M$, then the 2-cells bounded by $C'_i \cup C''_i$ and $C'_j \cup C''_j$ are mutually exclusive and the 2-cells bounded by $C'_i \cup C''_i$ does not intersect $F_n(B(n))$. Let $M = \{i(1), i(2), \ldots, i(t)\}$. For each $i \in M$, let $h_i$ be a diffeomorphism which is the identity on the complementary domain of $C'_i \cup C''_i$ and so that $h_i$ takes the 2-cells bounded by $C'_i \cup C_i$ into $\text{Int} F_n(B(n+1))$. Let $h = h_i(1) \circ h_i(2) \circ \cdots \circ h_i(t)$ and let $F_{n+1} = h^{-1} \circ F_n$.

**Notation.** Throughout Lemma 2 and Theorem 3, we let
H = \{(0,y) | y \leq 0\}.

Definition. We will say that the set K is enveloped by the open set U if K \subseteq \text{int} U.

Lemma 3. Suppose that \{U(j)\} \in J is a sequence of open and connected subsets of \mathbb{R}^2, cl U(j+1) \subseteq U(j) and \bigcap_{j \in \omega} U(j) = \emptyset. Suppose further that:

A. \{p(j)\} \in J is a sequence of points so that p(j) \in U(n) with \{|p(j)|\} \in J increasing and unbounded.

B. \{N(j)\} \in J is a family of disjoint, infinite subsets of \omega.

Then there is a diffeomorphism g of \mathbb{R}^2 onto an open subset of \mathbb{R}^2 such that

1. \mathbb{R}^2 - g(\mathbb{R}^2) is H.
2. each point of H is a limit point of \{g(p(n))| n \in N(j)\} for each \(j \in \omega.
3. g(U_n) envelopes H for each n \in \omega.

Proof. We construct G in several steps:

Step 1. Let h_0 be a diffeomorphism from \{(x,0) | x \in \mathbb{R}\} into \mathbb{R}^2 so that h_0(n,0) = p(n) and h_0(\{(x,0) | x > n\}) \subseteq U(n). Let h_1 be the extension of h_0 taking \mathbb{R}^2 onto \mathbb{R}^2 given by Lemma 2. Let h = h_1^{-1}.

Step 2. Let f be a diffeomorphism from \mathbb{R}^2 onto \mathbb{R}^2 which leaves the set \{(x,0) | x \geq 0\} fixed and so that \{(x,y) | x > n\} \subseteq f(h(U(n))).

Step 3. Let S = \{s_i | i \in \omega\} be a countable dense subset of \mathbb{R}. Let \phi be a diffeomorphism from \mathbb{R}^2 into \mathbb{R}^2 so that
(a) \( \phi(x,y) = (x,y') \) (i.e., \( \phi \) is fixed on its first coordinate).

(b) If \( N(j) = \{j(1), j(2), \cdots\} \), then \( \phi(j(i) + 1, 0) = (j(i) + 1, s_i) \).

Thus, \( j(i) \) is the \( i \)th number in \( N(j) \) and \( \phi \circ f \circ h \) takes \( p(j(i)) \) onto \( (j(i) + 1, s_i) \).

**Step 4.** Let \( \beta: \mathbb{R}^2 \to \mathbb{R}^2 \) be defined by \( \beta(x,y) = (e^{-x}, y) \).

**Step 5.** Let \( \gamma: \{(x,y)|x > 0\} \to \mathbb{R}^2 - H \) be defined by

\[
\gamma(x,y) = (\sqrt{x^2 + y^2} \cos (\pi/2 + 2 \arctan (y/x)), \sqrt{x^2 + y^2} \sin (\pi/2 + 2 \arctan (y/x))).
\]

Finally \( g = \gamma \circ \beta \circ \phi \circ f \circ h \) is the desired diffeomorphism.

**Theorem 2.** Assuming the continuum hypothesis, there is a hereditarily separable, perfectly normal, analytic manifold that is not metrizable.

**Proof.** We will build a \( C^\infty \)-manifold; the existence of an analytic manifold will then follow from [K,P]. The construction is simply a "careful" version of the construction developed in [RZ]. Let \( D = D(1) = \{x \in \mathbb{R}^2 | |x| \leq 1\} \) and let \( D^0 = \text{int } D \). Let \( \{x_\alpha | \alpha \in \omega_1\} \) be an indexing of \( D - D^0 \) (using CH). Let \( \{H_\alpha | \alpha \in \omega_1\} \) be a collection of mutually exclusive copies of \( H \). Let \( X_0 = \mathbb{R}^2 \) and let \( X_\alpha = X_0 \cup (\bigcup_{\beta < \alpha} H_\beta) \) and using CH, let \( \{A_\alpha | \alpha \in \omega_1\} \) be an indexing of the countable subsets of \( X \) so that \( A_\alpha \subset X_\alpha \). Let \( f_0 \) be a diffeomorphism from \( \mathbb{R}^2 \) onto \( D^0 \) and let \( F \) be the function defined by

\[
f(x) = \begin{cases} 
  f_0(x) & \text{if } x \in \mathbb{R}^2 \\
  x_\alpha & \text{if } x \in H_\alpha
\end{cases}
\]

and let \( f_\alpha = f|X_\alpha \). We will inductively construct a
differentiable structure $\mathcal{D}_\alpha$ on $X_\alpha$ such that:

1. $(X_\alpha, \mathcal{D}_\alpha)$ is diffeomorphic to $\mathbb{R}^2$: i.e. $\mathcal{D}_\alpha$ contains a chart $(X_\alpha, \phi_\alpha)$ with $\phi_\alpha(X_\alpha) = \mathbb{R}^2$.

2. If $\beta < \alpha$, then $(X_\beta, \phi_\beta) \in \mathcal{D}_\alpha$.

3. If $\gamma \leq \beta < \alpha$, $x \in H_\beta$ and $x_\beta$ is a limit point of $f(A_\alpha)$ in $D$, then $x$ is a limit point of $A_\alpha$ in $(X_\alpha, T_\alpha)$, where $T_\alpha$ is the topology on $X_\alpha$ given by $\mathcal{D}_\alpha$.

Let $\mathcal{D}_0$ be the usual differential structure on $X_0 = \mathbb{R}^2$ generated by the atlas consisting of the single chart $(X_0, \text{identity map})$.

Suppose we have $\mathcal{D}_\alpha$ satisfying (1)-(3) for all $\alpha < \lambda < \omega_1$.

**Case I.** $\lambda$ is a limit ordinal: Let $\mathcal{D}_\lambda$ be the differential structure generated by $\{(X_\theta, \mathcal{D}_\theta) \mid \theta < \lambda\}$. That $(X_\lambda, \mathcal{D}_\lambda)$ is diffeomorphic to $\mathbb{R}^2$ is given by Theorem 1.

**Case II.** $\lambda = \alpha + 1$: For each $n \in \omega$, let $U_n = f^{-1}_\alpha(D_{1/n}(x_\alpha))$, where $D_{1/n}(x_\alpha) = \{x \in D \mid d(x, x_\alpha) < 1/n\}$.

Then $\{U_n\}$ is a nested sequence of open sets in $X_\alpha$ such that $\bigcap_{n \in \omega} U_n = \phi$. Let $\{N_j\}_{j \in \omega}$ be a disjoint family of infinite subsets of $\omega$ and fix a 1-1 map $i: \omega + 1 \rightarrow \omega$. For each $n \in \omega$, choose $p_n \in U_n$ so that if $\beta \leq \alpha$ and $x_\alpha$ is a limit point of $f(A_\beta)$ in $D$, then $p_n \in A_\beta \cap U_n$ for all $n \in N_i(\beta)$.

Let $\phi$ be the diffeomorphism from $(X_\alpha, \mathcal{D}_\alpha)$ onto $\mathbb{R}^2$ given by our induction and let $g$ be the diffeomorphism given by Lemma 3 from $\mathbb{R}^2$ into $\mathbb{R}^2$ so that (1) $\mathbb{R}^2 - g(\mathbb{R}^2)$ is $H$, (2) each part of $H$ is a limit point of $\{g(\phi(p(k))) \mid k \in N_j\}$ for each $j \in \omega$, and (3) $g(U_n)$ envelopes $H$ for each $n \in \omega$. Let $\mathcal{D}_{\alpha+1}$ be the differential structure on $X_{\alpha+1}$ generated by the
atlas $\mathcal{D}_a \cup \{(X_{a+1}, \phi_{a+1})\}$ where $\phi_{a+1}|_{X_a} = g \circ \phi_a|_{\mathcal{H}_a}$ is the identification of $\mathcal{H}_a$ with $H$.

As in [RZ], the construction of $\mathcal{D}_{a+1}$ is such that $f_{a+1}$ is continuous and our induction is complete. We will let $\mathcal{D}$ be the atlas on $X$ generated by $\bigcup_{a<\omega_1} \mathcal{D}_a$ and let $T$ be the topology on $X$ given by $\mathcal{D}$. The argument that $(X,T)$ is hereditarily separable, perfectly normal, but not Lindelöf follows exactly as in [R,Z].

Note. As with the Rudin-Zenor manifold, we can, using $\phi$, obtain a differentiable, perfectly normal, countably compact, hereditarily separable, non-metrizable manifold. It remains an open question if there is a complex analytic, perfectly normal, non-metrizable manifold.

References


