Research Announcement:

THE SPAN OF MAPPINGS AND SPACES

by

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Let $X$, $Y$ be metric spaces, and let $f: X \to Y$ be a mapping. By $p_1$ and $p_2$ we denote the standard projections of the product $X \times X$ onto $X$, i.e., $p_1(x,x') = x$ and $p_2(x,x') = x'$ for $(x,x') \in X \times X$. The span $\sigma(f)$ of the mapping $f$ is the least upper bound of the set of real numbers $\alpha$ with the following property: there exist connected sets $C_\alpha \subseteq X \times X$ such that $p_1(C_\alpha) = p_2(C_\alpha)$ and $\alpha < \text{dist}[f(x),f(x')]$ for $(x,x') \in C_\alpha$ (see [2], p. 99). The span $\sigma(X)$ of the space $X$ is the span of the identity mapping on $X$ (see [4], p. 209). The purpose of the present paper (1) is to announce some results which relate to spans of mappings and have a number of interesting consequences for spans of spaces. A complete version will be published elsewhere.

The proofs of the following four propositions are rather straightforward.

1. If $f: X \to Y$, then $0 \leq \sigma(f) \leq \sigma(Y) < \text{diam } Y$.

2. If $f: X \to Y$ and $X$ is compact, then
   \[ \inf \{d[f^{-1}(y),f^{-1}(y')] : \sigma(f) < \text{dist}(y,y')\} \leq \sigma(X). \]

3. If $f: X \to Y$, $X$ is compact and $0 < \epsilon \leq \text{diam } Y$, then
   \[ 0 < \inf \{d[f^{-1}(y),f^{-1}(y')] : \epsilon \leq \text{dist}(y,y')\}. \]

4. If $f: X \to Y$ and $X$ is compact, then $\sigma(X) = 0$ implies $\sigma(f) = 0$.

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Note that proposition 4 follows from propositions 2 and 3. By \( S \) we denote the unit circle on the plane, and by \( T \) we denote the union of two tangent circles each of radius \( 1/2\pi \). We consider \( T \) to be a metric space with the geodesic metric \( \rho \). In other words, \( \rho(y,y') \) is the length of the shortest arc joining the points \( y \) and \( y' \) in \( T \) for \( y,y' \in T \), so that the diameter of \( T \) is one. We say that a mapping is essential if it is not homotopic to a constant mapping.

5. **Lemma.** If \( f: S \to T \) is an essential mapping and \( 0 < \varepsilon < 1/2 \), then there exist a continuum \( K \) and two surjective mappings \( \phi, \psi: K \to S \) such that

\[
\rho[f\phi(x), f\psi(x)] = \varepsilon \quad (x \in K).
\]

6. **Theorem.** If \( f: X \to T \) is an essential mapping, \( X \) is compact, \( \dim X \leq 1 \) and \( 0 < \varepsilon < 1/2 \), then there exists a continuum \( K \subseteq X \times X \) such that \( \rho_1(K) = \rho_2(K) \) and \( \rho[f(x), f(x')] = \varepsilon \) for \( (x, x') \in K \).

The following four statements are corollaries to theorem 6.

7. If \( f: X \to T \) is an essential mapping, \( X \) is compact and \( \dim X \leq 1 \), then \( \sigma(f) \geq 1/2 \).

8. If \( f: X \to T \) is an essential mapping, \( X \) is compact, \( \dim X \leq 1 \) and \( 0 < \varepsilon \leq 1/2 \), then

\[
0 < \text{Inf} \{ d[f^{-1}(y), f^{-1}(y')] : \rho(y, y') = \varepsilon \} \leq \sigma(X).
\]

9. If \( X \) is compact and \( \sigma(X) = 0 \), then each mapping \( f: X \to T \) is inessential.
10. If $X$ is a continuum and $\sigma(X) = 0$, then $X$ is tree-like.

It is known [4] that continua of span zero are one-dimensional if non-degenerate. By corollary 9, the mappings defined on them and having values in one-dimensional polyhedra [3] are all inessential, and then corollary 10(2) can be obtained via a well-known characterization of tree-like continua [1]. Also, notice that $\sigma(T) = \frac{1}{2}$. Hence, by proposition 1 and corollary 7, we get $\sigma(f) = \sigma(T)$ for all essential mappings $f$ of one-dimensional compact metric spaces into $T$. It remains as an open problem to determine a wider class of mappings $f: X \to Y$ such that $\sigma(f) = \sigma(Y)$.

References


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\[\text{A recent result of James F. Davis establishes the equality between the span and the semi-span [5] for a certain class of continua. Using the tree-likeness of continua of span zero (corollary 10), it implies, among other things, that they have the fixed point property.}\]