A NOTE ON INFINITE SEQUENTIAL ORDER

by

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I. Introduction

In this paper the example space $K_{\omega_0}$ of [1] is proven, in Theorem 2.2, to be the test space for regular sequential spaces of order $\omega_0$. A test space for sequential spaces of order $\alpha$, where $\alpha$ is an ordinal $\leq \omega_1$, is a space $T_\alpha$ such that if $X$ is any sequential space of order $\alpha$, then there is a subspace $Y$ of $X$ such that the sequential extension of $Y$ is homeomorphic to $T_\alpha$. The sequential extension [1] of a space $X$ is denoted $s(X)$ and is defined to be the set $X$ retopologized by letting all of the sequentially open sets be open. The Lemma 2.1, which was established to prove this theorem, provides a procedure for selecting a sequence of subspaces $T_i$ such that $T_i \subseteq X - \bigcup \{T_k : k < i\}$, $\sigma(T_i) > i$ and $\sigma(X - \bigcup \{T_k : k < i\}) = \omega_0$. This lemma appears to be a promising approach to prove test space theorems for all regular sequential spaces whose sequential order is a limit ordinal. The rather fascinating fact discovered here is that this lemma is false for all ordinals $\beta \geq \omega_0 + 1$, which are the sum of two smaller ordinals, as Example 2.3 shows. These examples show again the unpredictable behavior of the ordinal invariant $\sigma$, even on closed or open subspaces. The question of whether the spaces $K_{\beta}$, $\beta < \omega_1$ are the test spaces for regular sequential spaces whose order is greater than
$\omega_0$ is answered negatively here by a transfinite extension of the example of V. Kannan [4, Ex. 1.2, page 197].

Kannan has provided a retopologized variation of $K_{\omega_0+1}$ denoted $B_1$, which has sequential order $\omega_0+1$ and does not contain a copy of $K_{\omega_0+1}$. Also, $K_{\omega_0+1}$ does not contain a copy of $B_1$. Hence neither $K_{\omega_0+1}$ nor $B_1$ can be test spaces for spaces of sequential order $\omega_0+1$. Transfinite extensions of Kannan's example are constructed in Example 2.5 providing spaces of order $\alpha$ for all $\omega_0 < \alpha < \omega_1$ for which the spaces $K_{\alpha}$ are not test spaces in the sense of sequential extensions. Hence this brief paper presents the sharpest possible theorem related to test spaces for sequential spaces of infinite order, in the sense that $\sigma(X) \geq \beta$ implies the existence of a subspace $V_\beta$ whose sequential extension is a space of a given type. It is a continuation of studies from [1], [2], [3], and [4] and the reader is referred to [1] or [2] for the necessary definitions and notation.

II. Results

Complement Lemma 2.1. Let $\sigma(X) = \alpha$ where $\alpha$ is not the sum of two ordinals strictly less than $\alpha$. If $A$ is an open or closed subspace with $\sigma(A) = \beta < \alpha$, then $\sigma(X - A) = \alpha$.

Proof. Let $A$ be closed. Assume $\sigma(X - A) = \gamma < \alpha$. Since $\sigma(X) = \alpha$, there exists a set $B \subset X$ such that $B^\mu \neq \text{cl}_X(B)$ for $\mu > \gamma + 1 + \beta$. Let $p \in B^{\mu+1} - B^\mu$. Let $C = B - A$ and $T = X - A$. Suppose $p \in T$. Since $C^\gamma_T \subset B^\mu$
and \( p \notin B^\mu \), \( p \notin C_T^\gamma \). Since \( C_T^\gamma \) is closed in \( T \), there exists a neighborhood \( U \) of \( p \) such that \( U \subseteq T \) and \( U \cap C_T^\gamma = \emptyset \). Since \( T \) is open \( U \cap C_T^\gamma = U \cap (C^\gamma \cap T) = U \cap C^\gamma = \emptyset \). Thus \( U \cap (B - A) = \emptyset \) and \( U \cap (B \cap A) = \emptyset \). Hence we have the contradiction \( U \cap B = \emptyset \). Accordingly, \( p \in A \). Since \((B \cap A)^\beta \) is closed in \( A \) and thus closed in \( X \) and \((B \cap A)^\beta \subset B^\mu \), \( p \notin (B \cap A)^\beta = cl_X(B \cap A) \). Thus there exists a neighborhood \( U \) of \( p \) such that \( U \cap A = \emptyset \). Since \( p \in B^{\mu+1} - B^\mu \), \( U \cap B^\mu \neq \emptyset \). Since \( p \notin cl_X(B \cap A) \), \( p \in cl_X(B - A) = cl_X(C) \). Say \( p \in C^\nu \). Since \( C^\gamma \subset B^\gamma \subset B^\mu \), \( p \notin C^\nu \) and \( \nu > \gamma \). Since \( \sigma(X - A) = \gamma \), \((C^{\gamma+1} - C^\gamma) \cap T = \emptyset \). Thus \( C^{\gamma+1} - C^\gamma \subset A \) and since \( A \) is closed \( C^{\delta+1} - C^\delta \subset A \) for each \( \delta > \gamma \). Since \( p \in C^\nu \) for some \( \nu > \gamma \) and \( p \notin C^\gamma \), \( p \in cl_X(C^{\gamma+1} - C^\gamma) \). Since \( C^{\gamma+1} - C^\gamma \subset A \) and \( A \) is closed and \( \sigma(A) = \beta \), \( p \in cl_X(C^{\gamma+1} - C^\gamma) = (C^{\gamma+1} - C^\gamma)^\beta \subset C^{\gamma+1+\beta} \). However, \( C^{\gamma+1+\beta} \subset B^{\gamma+1+\beta} \subset B^\mu \), because \( \mu > \gamma + 1 + \beta \) and this implies the contradiction, \( p \in B^\mu \). Thus \( \sigma(X - A) = \alpha \) and this completes the proof for the case where \( A \) is a closed set. Suppose \( A \) is an open set. Then \( X - A \) is closed and the assumption that \( \sigma(X - A) = \gamma < \alpha \) results in the same contradiction from the preceding proof. This completes the proof.

The preceding (complement) lemma was established for the case \( \alpha = \omega_\alpha \), to prove Theorem 2.2. The extension to all ordinals which are not the sum of smaller ordinals is the result of an inquiry by the referee. This lemma provides a means of successively selecting a closed or
open subspace from the remaining space whose sequential order is sufficiently large.

The existence of the open subspaces of desired order is a consequence of proof of Proposition 3.1 in [1] and is established as follows. For any sequential space $Y$ where $\sigma(Y) = \omega_0$, for each $n < \omega_0$ there exists a subspace $Y'$ selected from the sequential closures of a set $A$ such that $s(Y') = S_n$. Thus if $y$ is the base point of $Y'$ and $p$ is any fixed point in the sequence in $Y'$ converging to $y$ there are disjoint open sets $H_{n-1}$ and $H_n$ such that $p \in H_{n-1}$ and $y \in H_n$. Hence $A \cap H_{n-1}$ and $A \cap H_n$ are subsets of $H_{n-1}$ and $H_n$ respectively which require at least $n-1$ and $n$ sequential closures in $H_{n-1}$ and $H_n$ and thus $\sigma(H_{n-1}) > n-1$ and $\sigma(H_n) > n$. Thus, if $\sigma(Y) = \omega_0$, for each $n < \omega_0$ there are disjoint open subspaces $H_{n-1}$ and $H_n$ such that $\sigma(H_{n-1}) > n-1$ and $\sigma(H_n) > n$.

Theorem 2.2. If $X$ is a regular sequential space and $\sigma(X) = \omega_0$, then $X$ contains a subspace $T$ such that $s(T) = K_{\omega_0}$.

Proof. Let $\sigma(X) = \omega_0$. There exist open subspaces $V_1$ and $U_2$ such that $V_1 \cap U_2 = \emptyset$, $\sigma(V_1) > 1$ and $\sigma(U_2) > 1$ and thus $\sigma(X - V_1) > 1$. If $\sigma(V_1) = \omega_0$, then let $T_1$ be an $S_1$ in $U_2$ and let $G_1 = U_2$. If $\sigma(V_1) = m < \omega_0$, by the complement lemma, $\sigma(X - V_1) = \omega_0$. Then let $T_1$ be an $S_1$ in $V_1$ and let $G_1 = V_1$. Let $X_1 = X - G_1$. Assume this process has been repeated $n-1$ times. That is, for each $k \leq n-1$, $X_k = X_{k-1} - G_k$, where $G_k$ is open in $X_{k-1}$, $T_k$ is a subspace of $G_k$ such that $s(T_k) = S_k$ and $\sigma(X_k) = \omega_0$. Since
\[ \sigma(X_{n-1}) = \omega_o, \] there exists open subspaces \( V_n \) and \( U_{n+1} \) of \( X_{n-1} \) such that \( V_n \cap U_{n+1} = \emptyset, \sigma(V_n) \geq n, \sigma(U_{n+1}) > n \) and thus \( \sigma(X_{n-1} - V_n) > n. \) If \( \sigma(V_n) = \omega_o, \) then let \( T_n \) be a subspace of \( U_{n+1} \) such that \( s(T_n) = S_n \) and let \( G_n = U_{n+1}. \)

If \( \sigma(V_n) = m < \omega_o, \) then let \( T_n \) be a subspace of \( V_n \) such that \( s(T_n) = S_n \) and let \( G_n = V_n. \) By the complement lemma,

\[ \sigma(X_{n-1} - V_n) = \omega_o. \] Let \( X_n = X_{n-1} - G_n. \) Thus, \( G_n \) is an open subspace of \( X_{n-1}, \) \( T_n \) is a subspace of \( G_n \) such that \( s(T_n) = S_n \) and \( \sigma(X_n) = \omega_o. \) This completes the induction step and for each \( n < \omega_o, \) \( T_n \) is a subspace of \( G_n, \) such that \( s(T_n) = S_n. \) For each \( n < \omega_o, \) let \( p_n \) be the base point of \( T_n \) and let \( H = \{ p_n : n < \omega_o \}. \) If \( H \) has no cluster point, then since \( X \) is regular there is a disjoint collection of open subsets of \( X, \{ C_n : n < \omega_o \} \) such that \( p_n \in C_n, \) for each \( n < \omega_o. \) If \( H \) has cluster point there is a subsequence of \( \{ p_n \} \) that converges to some point. Thus there is in this case a disjoint collection of open sets in \( X \) each containing exactly one point of the convergent subsequence. Hence in either case, there is a sequence of base points \( \{ p_k \} \) and a disjoint collection of open sets \( U_k \) such that \( p_k \in U_k, \) for \( k < \omega_o. \) Let \( T'_k = U_k \cap T_k \) for each \( k. \) Then since the sequential extension of \( T'_n, s(T'_n), \) is \( S_n \) for each \( n, \) \( s(\cup\{ T'_n : n < \omega_o \}) = \cup\{ s(T'_n) : n < \omega_o \} = K_{\omega_o}. \) Accordingly, \( T = \cup\{ T'_n : n < \omega_o \} \) is a subspace of \( X \) such that \( s(T) = K_{\omega_o}. \) This completes the proof.

The following examples establish the sharpness of the results in Theorems 2.1 and 2.2. That is, the Complement
Lemma 2.1 is false for all infinite ordinals which are the sum of smaller ordinals and for each ordinal $\alpha$, $\omega_0 < \alpha < \omega_1$, there is a regular sequential space of order $\alpha$ for which $K_\alpha$ is not a test space using sequential extensions.

**Example 2.3.** For every ordinal $\alpha$, $\omega_0 < \alpha < \omega_1$, which is the sum of two ordinals $\beta$ and $\gamma$, $\alpha = \beta + \gamma$, where $\beta < \alpha$ and $\omega_0 < \gamma < \alpha$, there is a regular sequential space $X$, such that $\sigma(X) = \alpha$, which has an open subspace $A$ such that $\sigma(A) = \beta + 1 < \alpha$ and $\sigma(X - A) = \gamma < \alpha$.

For each isolated point, $y$, of $K_\gamma$, let $K_{\beta+1}(y)$ be a copy of $K_{\beta+1}$ with base point $0_y$. Form the quotient space $X$ by attaching the base point $0_y$ of $K_{\beta+1}(y)$ to the isolated point $y \in K_\gamma$. Then $\sigma(X) = \alpha$, $A = \bigcup\{K_{\beta+1}(y) : y \text{ is isolated in } K_{\gamma}\}$ is an open subspace of $X$ with $\sigma(A) = \beta + 1 < \alpha$ and $\sigma(X - A) = \gamma < \alpha$.

**Example 2.4.** There is a sequential space $T$ where $\sigma(T) = \omega_0+1$ and an open subspace $A$ of $T$ such that $\sigma(A) < \omega_0+1$ and $\sigma(T - A) < \omega_0+1$.

Let $T$ be the space $K_{\omega_0+1}$ and let $A$ be the sequence in $T$ converging to the base point of $K_{\omega_0+1}$. Then $\sigma(A) = 1 < \omega_0+1$ and since $T - A$ is the disjoint union of the spaces $S_n$, for $n < \omega_0$, $\sigma(T - A) = \omega_0 < \omega_0+1$.

The following example supplies, in two ways, examples of spaces of order $\alpha$ for which $K_\alpha$ is not a test space, under sequential extensions, for all ordinals $\alpha$ such that $\omega_0 < \alpha < \omega_1$. 
Example 2.5. For each $\alpha$, $\omega_0 < \alpha < \omega_1$, there is a sequential space $X_\alpha$ such that $\sigma(X_\alpha) = \alpha$ which does not contain a subspace whose sequential extension is $K_\alpha$.

The construction of the spaces $X_\alpha$ is by induction on the non-limit ordinals between $\omega_0$ and $\omega_1$. The sequential order at a point $p$ in a space $X$ is defined as $\sigma(p,X) = \inf(\alpha : p \in B^\alpha$, for all $B \subseteq X$ with $p \in \text{cl}_X(B))$. For each ordinal $\eta$ let $V_\eta = \{x \in X : \sigma(x,X) \geq \eta\}$. For $\alpha = \omega_0 + 1$ let $X^{\omega_0 + 1}$ be the space $K^{\omega_0 + 1}$ retopologized only at the base point $0$ in the following way. A neighborhood of $0$ is a set $V \subseteq K^{\omega_0 + 1}$ such that $0 \in V$, there exists $n < \omega_0$ such that $V_n \subseteq V$ and $V - \{0\}$ is open as a subset of $K^{\omega_0 + 1}$. Then $X^{\omega_0 + 1}$ is a sequential space, $\sigma(X^{\omega_0 + 1}) = \omega_0 + 1$ and neither $X^{\omega_0 + 1}$ nor $K^{\omega_0 + 1}$ can be embedded in the other, because of the neighborhoods of $0$. Suppose $X_\alpha$ has been defined for all non-limit ordinals $\alpha < \beta = \gamma + 1$. In the case where $\gamma$ is a limit ordinal, choose an increasing sequence of non-limit ordinals $\beta_i \to \gamma$. Form the space $X_\beta$ by attaching the base point $0_1$ of $X_{\beta_i}$ to $\frac{1}{1}$ in $S_1$ for each $i$. Let a nhhood of the base point $0$ (from $S_1$) be a set $V$ such that $0 \in V$, there exists some $\alpha < \beta$ such that $V_\alpha \subseteq V$ and $V - \{0\}$ is open in the space $X_\beta - \{0\}$. (This is the disjoint topological sum of the spaces $V_{\beta_i}$, $i < \omega_0$.) In the case where $\gamma$ is a non-limit ordinal, for each $i < \omega_0$ let $X_{\gamma}(i)$ be a copy of $X_\gamma$ with base point $0_1$. Form the quotient space $X_\beta$ by attaching the base point $0_1$ of $X_\gamma(i)$ to
in $S_i$ for each $i$. Then in either case $X_\beta$ is a sequential space, $\sigma(X_\beta) = \beta$. ($X_\beta$ is $K_\beta$ retopologized at each point of infinite order.) Neither $X_\beta$ nor $K_\beta$ can be embedded in the other because of the neighborhoods of the points of infinite order. For the countable limit ordinals $\beta$, let $X_\beta$ be the disjoint topological sum of the spaces $X_\alpha$, $\alpha < \beta$. This completes the construction of the spaces $X_\alpha$, $\omega_0 < \alpha < \omega_1$. These examples are rather extreme in the sense that the topology is drastically altered at every point of infinite order in $X_\alpha$. Another way of building a collection of spaces for which the $K_\alpha$ spaces do not suffice as test spaces can be described as follows. Let $Y_{\omega_0+1} = X_{\omega_0+1}$ from before. Let $\alpha$ be any non-limit ordinal, $\omega_0+1 < \alpha < \omega_1$. Let $Y_\alpha$ be the space $K_\alpha$ with the neighborhoods of only the points of order $\omega_0+1$ altered to have a neighborhood base as in $Y_{\omega_0+1}$. Since any neighborhood of the base point in $Y_\alpha$ must contain infinitely many of the points of order $\omega_0+1$, for the reasons stated before the $K_\alpha$ spaces can not be the test spaces for the spaces $Y_\alpha$ either.

We would like to express our appreciation to the referee whose questions caused the original version of this paper to be substantially improved.

References


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