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By a *linearly ordered topological space* (LOTS) we mean a linearly ordered set equipped with the usual open interval topology of the given order. By a *generalized ordered space* (GO-space) we mean a linearly ordered set equipped with a T_1-topology for which there is a base of order-convex sets [L]. To say that a topological space is *perfect* means that every closed subset of the space is a G_δ-set. Finally, for any space X, the set of non-isolated points of X is denoted by X^d.

In abstract spaces, the property of being perfect has little relationship to other familiar properties. This contrasts with the situation in ordered spaces where, for example, it is known that a separable GO-space must be perfect, and a perfect GO-space must be paracompact [BL_1] [EL]. In [vW], van Wouwe sharpened that first implication by proving

1.1. *Theorem. If a* GO-*space has a* σ-*discrete dense subspace, then it is perfect.*

Maarten Maurice has asked whether the converse of van Wouwe's theorem is valid, provided there are no Souslin spaces. In any model of set theory where Maurice's question has an affirmative answer (and it is not clear that such models exist since, for all we know, there may be an

Proofs of the Results

2.1. *Lemma. A* GO-*space* X *is perfect if and only if*

(a) X *is first countable; and*

(b) every pairwise-disjoint collection of open convex sets is σ-*locally finite collection.*

2.2. *Proof of Theorem* 1.2. Let I be the set of isolated points of X. Then I is an F_σ-subset of X since X is perfect. Find a sequence X(n) of closed nowhere dense subsets of X having $X(n) \subset X(n+1)$ and $X^d = \cup\{X(n): n \geq 1\}$. Each set X - X(n) is open and therefore is an F_σ-set, say $X - X(n) = \cup\{F(n,k): k \geq 1\}$, where $F(n,1) \subset F(n,2) \subset \cdots$ and each F(n,k) is a closed subset of X. Let $\mathcal{C}(n,k)$ be the family of convex components of the set X - F(n,k). Write the open set X - F(n,k) as $\cup\{E(n,k,j): j \geq 1\}$ where the sets E(n,k,j) are closed and satisfy $E(n,k,1) \subset E(n,k,2) \subset \cdots$. Let $\mathcal{C}(n,k,j) = \{J \in \mathcal{C}(n,k): J \cap E(n,k,j) \neq \emptyset\}$. The collection $\mathcal{C}(n,k,j)$ is locally finite (cf. (2.1)) and pairwise disjoint so that if we choose one point $d(J,n,k,j) \in J \cap E(n,k,j)$ for each $J \in \mathcal{C}(n,k,j)$, the resulting set $D(n,k,j) = \{d(J,n,k,j): J \in \mathcal{C}(n,k,j)\}$ will be a closed discrete subset of X. Then the set $D = I \cup (\cup\{D(n,k,j): n,k,j \geq 1\})$ is a σ-discrete subset of X. We claim that D is dense in X. For suppose U is a nonempty open subset of X. If $U \cap I \neq \emptyset$ there is nothing to prove, so assume $U \subset X^d$. Choose points p < q of X such that $\emptyset \neq]p,q[\subset U$. Since]p,q[must be infinite, we may find points r_1,r_2,r_3 having $p < r_1 < r_2 < r_3 < q$. Then there is an index n_0 so large that $\{r_1, r_2, r_3\} \subset X(n_0)$. Since $X(n_0)$ is nowhere dense, neither $]p, r_2[$ nor $]r_2, q[$ can be subsets of $X(n_0)$. Choose k_0 so large that both $]p, r_2[$ and $]r_2, q[$ meet $F(n_0, k_0)$, and choose points $s_1 \in]p, r_2[\cap F(n_0, k_0)$ and $s_2 \in]r_2, q[\cap F(n_0, k_0)$. Since $r_2 \in X(n_0)$, some convex component J_0 of $X - F(n_0, k_0)$ contains r_2. Choose j_0 so large that $J_0 \in \mathcal{C}(n_0, k_0, j_0)$. Since J_0 is convex, meets $]s_1, s_2[$ and contains neither s_1 nor s_2, we have $J_0 \subset]s_1, s_2[\subset]p, q[\subset U$ so that the point $d(J_0, n_0, k_0, j_0) \in D \cap U$.

2.3. *Proof of* (1.3). If Y is a subspace of a Souslin space X, then Y is a hereditarily Lindelöf GO-space [BL_1]. If Y were first category in itself, then Theorem 1.2 would yield a σ-discrete dense subset $D \subset Y$. But then D would be countable so that Y would be separable, contrary to hypothesis. To prove the second assertion of (1.3), recall that any space X is a Baire space if and only if each open subset of X is second category in itself. If we assume that each open interval in our Souslin space is non-separable, then the first assertion of (1.3) applies to yield the desired conclusion.

2.4. *Proof of* (1.4). Since X is a perfect LOTS which is first category in itself, X has a σ-discrete dense subset. Now the proof of Proposition 3.4 in [BL_2, p. 380] may be used to construct a σ-disjoint base for X.

Since X is perfect and paracompact, that is enough to force X to be metrizable.

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