ON A THEOREM OF CHABER

by

ROBERT L. BLAIR
ON A THEOREM OF CHABEL\textsuperscript{1,2}

Robert L. Blair

1. Introduction

For $\mathcal{S}$ a collection of subsets of a topological space $X$ and $x \in X$, set $S(x) = \{ S \in \mathcal{S} : x \in S \}$, $I(x, \mathcal{S}) = \cap S(x)$, $st(x, \mathcal{S}) = \cup S(x)$, and $ord(x, \mathcal{S}) = |S(x)|$. (|E| denotes the cardinal of the set E. Cardinals are initial ordinals.)

The following theorem is due to Chaber:

1.1. Theorem (Chaber [6, 3.B]). Let $\mathcal{U}$ be an open cover of a countably compact space $X$. If there exists an open cover $\bigcup_{n<\omega} \mathcal{G}_n$ of $X$ such that, for every $x \in X$, $\cap \{I(x, \mathcal{G}_n) : n < \omega \text{ and } 0 < ord(x, \mathcal{G}_n) = \omega\} \subseteq \bigcup$ for some $\mathcal{U} \in \mathcal{U}$, then $\mathcal{U}$ contains a finite subcover.

In this note we first prove a theorem (2.4) that quickly yields 1.1, and then obtain several results closely related to 1.1. Some of the latter generalize the main results of [3]. All of our results have cardinal generalizations, but for simplicity only the countable versions of these more general theorems will be considered here.

2. Closed-Completeness of $\delta\theta$-Penetrable Spaces

To state our results succinctly, we shall say that an open cover $\bigcup_{n<\omega} \mathcal{G}_n$ of a topological space $X$ is a $\theta$-penetration (resp. $\delta\theta$-penetration) of a cover $\mathcal{U}$ of $X$ if, for every

\textsuperscript{1}This research was supported in part by Ohio University Research Committee Grant No. 535.

\textsuperscript{2}Dedicated to Casper Goffman on his 66th birthday.
x ∈ X, \(\bigcap \{I(x, \mathcal{G}_n) : n < \omega \text{ and } 0 < \text{ord}(x, \mathcal{G}_n) < \omega\} \subset U\) for some \(U \in \mathcal{U}\) (resp. \(\bigcap \{I(x, \mathcal{G}_n) : n < \omega \text{ and } 0 < \text{ord}(x, \mathcal{G}_n) \leq \omega\} \subset U\) for some \(U \in \mathcal{U}\), and that \(X\) is \(\theta\)-penetrable (resp. \(\delta\theta\)-penetrable) if every open cover of \(X\) has a \(\theta\)-penetration (resp. \(\delta\theta\)-penetration).

2.1. Remarks. (a) A cover \(\mathcal{G}\) of \(X\) is separating if for each \(x, y \in X\) with \(x \neq y\) there exists \(G \in \mathcal{G}\) with \(x \in G\) and \(y \notin G\); and a cover \(\bigcup_{n<\omega} \mathcal{G}_n\) of \(X\) is \(\theta\)-separating [12, 3.1] if for every \(x \in X\), \(\bigcap\{I(x, \mathcal{G}_n) : n < \omega \text{ and } 0 < \text{ord}(x, \mathcal{G}_n) < \omega\} = \{x\}\). Obviously each point-countable separating open cover of \(X\) is a \(\delta\theta\)-penetration of every cover of \(X\), and each \(\theta\)-separating open cover of \(X\) is a \(\theta\)-penetration of every cover of \(X\).

(b) A weak \(\theta\)-refinement (resp. weak \(\delta\theta\)-refinement) of a cover \(U\) of \(X\) is an open refinement \(U_{n<\omega} \mathcal{G}_n\) of \(U\) such that \(X = \bigcup_{n<\omega} \{x \in X : 0 < \text{ord}(x, \mathcal{G}_n) < \omega\}\) (resp. \(X = \bigcup_{n<\omega} \{x \in X : 0 < \text{ord}(x, \mathcal{G}_n) \leq \omega\}\)) (see [2] and [18]). It is easily seen that (*) every weak \(\theta\)-refinement (resp. weak \(\delta\theta\)-refinement) of \(U\) is a \(\theta\)-penetration (resp. \(\delta\theta\)-penetration) of \(U\). The converse of (*), however, is false: Let \(X\) be an hereditarily separable non-Lindelöf space obtained by refining the usual topology of \(\mathbb{R}\) (see [11] and [14]); it suffices to observe that \(\mathbb{R}\) (and hence \(X\)) has a \(\theta\)-separating open cover but that, by [3, 3.16], \(X\) is not weakly \(\delta\theta\)-refinable (i.e. some open cover of \(X\) has no weak \(\delta\theta\)-refinement). But (*) has a partial converse; this is the substance of 2.2 below.
By a closed ultrafilter on $X$ we mean a maximal filter in the lattice of closed subsets of $X$. A closed ultrafilter $\mathcal{J}$ on $X$ is countably complete if $\bigcap A \in \mathcal{J}$ for every $A \subset X$ with $|A| \leq \omega$, and $\mathcal{J}$ is fixed (resp. free) if $\bigcap \mathcal{J} \neq \emptyset$ (resp. $\bigcap \mathcal{J} = \emptyset$). A space $X$ is closed-complete (= a-real-compact [7]) if every countably complete closed ultrafilter on $X$ is fixed. (If "closed" is replaced by "Borel" in the preceding definitions, one obtains the definition of a Borel-complete space; see [10] and [3, p. 20]. We note that Borel-completeness implies closed-completeness [10, 1.1].)

2.2. Lemma. If $\mathcal{J}$ is a countably complete free closed ultrafilter on $X$ and if $\mathcal{G} = \bigcup_{n<\omega} \mathcal{G}_n$ is a $\theta$-penetration (resp. $\delta\theta$-penetration) of $\mathcal{U} = \{X - F : F \in \mathcal{J}\}$, then $\mathcal{G}$ has a subcover that is a weak $\theta$-refinement (resp. weak $\delta\theta$-refinement) of $\mathcal{U}$.

Proof. We write the proof for the case in which $\mathcal{G}$ is a $\theta$-penetration of $\mathcal{U}$. For each $n < \omega$, let

$$A_n = \{x \in X : \text{ord}(x, \mathcal{G}_n) < \omega \text{ and } X - G \in \mathcal{J} \text{ for some } G \in \mathcal{G}_n(x)\}.$$ 

If there exists $y \in X - \bigcup_{n<\omega} A_n$, set

$$K = \{n \in \omega : 0 < \text{ord}(y, \mathcal{G}_n) < \omega\},$$

$$M = \{(n, G) : n \in K \text{ and } G \in \mathcal{G}_n(y)\}.$$ 

Then for each $(n, G) \in M$ we have $\text{ord}(y, \mathcal{G}_n) < \omega$, $G \in \mathcal{G}_n(y)$, and $y \notin A_n$, and thus $X - G \notin \mathcal{J}$; hence $P(n, G) \subseteq G$ for some $P(n, G) \in \mathcal{J}$. Then $\bigcap_{n \in K} (\bigcap_{G \in \mathcal{G}_n(y)} P(n, G)) \in \mathcal{J}$. But

$$\bigcap_{n \in K} (\bigcap_{G \in \mathcal{G}_n(y)} P(n, G)) \subseteq \bigcap_{n \in K} I(y, \mathcal{G}_n) \subseteq U.$$
for some \( U \in \mathcal{U} \), a contradiction, and we conclude that

\[ X = \bigcup_{n<\omega} A_n. \]

Now for each \( n < \omega \) and each \( x \in A_n \), there is \( G(x,n) \in \mathcal{G}_n(x) \) with \( G(x,n) \in \mathcal{U} \). For each \( n < \omega \), let \( \mathcal{G}_n* = \{ G(x,n) : x \in A_n \} \), and let \( \mathcal{G}^* = \bigcup_{n<\omega} \mathcal{G}_n^* \). Note that if \( x \in X \), then \( x \in A_n \) for some \( n \). Then \( x \in G(x,n) \in \mathcal{G}_n^* \) and

\[ |\mathcal{G}_n^*(x)| \leq |\mathcal{G}_n(x)| < \omega, \]

so \( \mathcal{G}^* \) is a weak 0-refinement of \( \mathcal{U} \).

2.3. Lemma (cf. [17, Chap. 1, Theorem 18]). Let

\( A \subseteq X \) and let \( \mathcal{G} \) be a collection of open subsets of \( X \) such that \( \text{cl} A \subseteq \bigcup \mathcal{G} \). Then there exists \( D \subseteq A \) such that:

1. If \( x, y \in D \) with \( x \neq y \), then \( x \notin \text{st}(y, \mathcal{G}) \).
2. \( A \subseteq \bigcup_{x \in D} \text{st}(x, \mathcal{G}) \).
3. \( \{ \text{cl}(x) : x \in D \} \) is discrete in \( X \).

Proof. By Zorn's lemma, there exists \( D \subseteq A \) maximal with respect to (1), and then \( D \) must satisfy (2). If \( \mathcal{D} = \{ \text{cl}(x) : x \in D \} \) is not discrete in \( X \), there is \( p \in \text{cl} A \) such that every neighborhood of \( p \) meets at least two distinct members of \( \mathcal{D} \). Then \( p \in G \) for some \( G \in \mathcal{G} \), so there exist \( x, y \in D \) with \( x \neq y \) such that \( G \cap \text{cl}(x) \neq \emptyset \) and \( G \cap \text{cl}(y) \neq \emptyset \). But then \( x \in \text{st}(y, \mathcal{G}) \), contrary to (1).

The discreteness character \( \Delta(X) \) of a space \( X \) is \( \omega^* \), where \( \kappa = \sup |\mathcal{D}| : \mathcal{D} \) is a discrete collection of nonempty closed subsets of \( X \) \([13, \S 3]\). (For a \( T_1 \)-space \( X \), \( \Delta(X) \) is the extent of \( X \) \([8, \text{1.7.12}] \) and \( X \) is \( \omega_1 \)-compact (i.e. every closed discrete subset of \( X \) is countable) if and only if \( \Delta(X) = \omega \) \([13, \text{3.2}] \).)

2.4. Theorem. If \( \Delta(X) = \omega \) and if \( \mathcal{J} \) is a free closed ultrafilter on \( X \) such that \( \{ X - F : F \in \mathcal{J} \} \) has a
\( \delta \)-penetration, then \( J \) is not countably complete.

Proof. If \( J \) is countably complete, then, by 2.2, \( \{ X - F : F \in J \} \) has a weak \( \delta \)-refinement \( \bigcup_{n<\omega} \mathcal{G}_n \), and there exists \( n < \omega \) such that \( A = \{ x \in X : 0 < \text{ord}(x, \mathcal{G}_n) \leq \omega \} \) meets every member of \( J \). Since \( A \subseteq \bigcup \mathcal{G}_n \), we have \( F^* \subseteq \bigcup \mathcal{G}_n \) for some \( F^* \in J \). By 2.3 there exists \( D \subseteq A \cap F^* \) with \( A \cap F^* \subseteq \bigcup_{x \in D} \text{st}(x, \mathcal{G}_n) \) and \( |D| \leq \Delta(X) = \omega \). Then \( W = \bigcup_{x \in D} \mathcal{G}_n(x) \) is countable, and for each \( W \in W \) there is \( F(W) \in J \) with \( W \subseteq X - F(W) \). But then \( A \cap F^* \cap (\bigcap_{W \in W} F(W)) = \emptyset \), a contradiction.

We obtain Chaber's theorem as follows:

Proof of 1.1. If the conclusion fails, then \( \{ X - U : U \in \mathcal{U} \} \subseteq J \) for some (free) closed ultrafilter \( J \) on \( X \), and by the hypothesis of 1.1, \( \{ X - F : F \in J \} \) has a \( \delta \)-penetration. But since \( X \) is countably compact, \( \Delta(X) = \omega \) and \( J \) is countably complete. This contradicts 2.4.

The following generalizes [3, 3.2]:

2.5. Corollary. If \( \Delta(X) = \omega \), then the following are equivalent:

(1) \( X \) is closed-complete.

(2) If \( J \) is any free closed ultrafilter on \( X \), then \( \{ X - F : F \in J \} \) has a \( \delta \)-penetration.

Proof. If \( X \) is closed-complete and \( J \) is a free closed ultrafilter on \( X \), then \( \bigcap_{n<\omega} F_n = \emptyset \) for some sequence \( (F_n)_{n<\omega} \) of members of \( J \), and clearly \( \bigcup_{n<\omega} \{ X - F_n \} \) is a \( \theta \)-penetration of \( \{ X - F : F \in J \} \). The converse is immediate from 2.4.
A space $X$ is *isocompact* [1] if every countably compact closed subset of $X$ is compact. We shall say that $X$ is *iso-closed-complete* (resp. *iso-Lindelöf*) if every closed subset of $X$ with countable discreteness character is closed-complete (resp. Lindelöf). Clearly every iso-Lindelöf space is iso-closed-complete, and since countably compact closed-complete spaces are compact [3, 3.6], every iso-closed-complete space is isocompact. Since $\delta\theta$-penetrability is closed-hereditary, Chaber's theorem evidently implies that $\delta\theta$-penetrable spaces are isocompact. More generally:

2.6. **Corollary.** Every $\delta\theta$-penetrable space is iso-closed-complete.

2.7. **Remarks.** The example of 2.1(b) shows that hereditarily $\theta$-penetrable regular $T_1$-spaces need not be iso-Lindelöf. For an example of an isocompact space that is not iso-closed-complete, let $X$ be the subspace of $\omega_2$ obtained by deleting all nonisolated points having a countable base (see [9, 9I]). Then every countably compact closed subset of $X$ is finite (so $X$ is isocompact), and $X$ is $\omega_1$-compact. But $X$ is normal, countably paracompact, and nonrealcompact, and thus not closed-complete [7, 1.10]. (This example was pointed out to the author by Eric van Douwen.)

It follows from 2.5 that an $\omega_1$-compact space with a point-countable separating open cover is closed-complete. But in this case a stronger result is available:
2.8. Theorem. If $X$ is an $\omega_1$-compact space with a point-countable separating open cover, then $X$ is Borel-complete.

Proof. Let $\mathcal{U}$ be a point-countable separating open cover of $X$ and let $\mathcal{J}$ be a countably complete Borel ultrafilter on $X$. Suppose that for each $x \in X$ there exists $U_x \in \mathcal{U}(x)$ with $X - U_x \in \mathcal{J}$. Let $\mathcal{G} = \{U_x : x \in X\}$. By 2.3 there exists $D \subseteq X$ with $X = \bigcup_{x \in D} \text{st}(x, \mathcal{G})$ and $|D| \leq \Delta(X) = \omega$. But then $\bigcup_{x \in D} \mathcal{G}(x)$ is a countable cover of $X$, which contradicts the countable completeness of $\mathcal{J}$. Thus there exists $x \in X$ such that $X - U \notin \mathcal{J}$ for every $U \in \mathcal{U}(x)$, and hence for every $U \in \mathcal{U}(x)$ there is $F(U) \in \mathcal{J}$ with $F(U) \subseteq U$. Then $\cap \{F(U) : U \in \mathcal{U}(x)\} \subseteq \cap \mathcal{U}(x) = \{x\}$, and since $\mathcal{U}(x)$ is countable, we have $\{x\} \in \mathcal{J}$. Thus $x \in \cap \mathcal{J}$.

3. Closed-Completeness of $\theta$-Penetrable Spaces

The lattice of closed subsets of a space $X$ is atomic if each nonempty closed subset of $X$ contains a minimal nonempty closed set. (This holds, for example, if $X$ is essentially $T_1$, i.e. for each $x$, $y \in X$, either $\text{cl}(x) \cap \text{cl}(y) = \emptyset$ or $\text{cl}(x) = \text{cl}(y)$.) The following generalizes [3, 4.1]:

3.1. Theorem. If the lattice of closed subsets of $X$ is atomic, then the following are equivalent:

(1) $X$ is closed-complete.

(2) The cardinal of each discrete collection of closed subsets of $X$ is Ulam-nonmeasurable, and if $\mathcal{J}$ is
any free closed ultrafilter on $X$, then \( \{X - F : F \in \mathcal{J} \} \) has a $\theta$-penetration.

**Proof.** (1) $\Rightarrow$ (2): Let $\mathcal{D}$ be a discrete collection of nonempty closed subsets of $X$; we may assume that each $D \in \mathcal{D}$ is minimal. For each $D \in \mathcal{D}$, choose $x_D \in D$, let $E = \{ x_D : D \in \mathcal{D} \}$, and let $\mathcal{E}$ be a countably complete ultrafilter on the (discrete) space $E$. Let $\mathcal{E}^* = \{ F : F \text{ is closed in } X \text{ and } F \cap E \in \mathcal{E} \}$. The minimality of the members of $\mathcal{D}$ allows one to conclude that $\mathcal{E}^*$ is a countably complete closed ultrafilter on $X$, and hence, by (1), there exists $y \in \cap \mathcal{E}^*$. Since $\cup \mathcal{D} \in \mathcal{E}^*$, $y \in D$ for some $D \in \mathcal{D}$, and it follows that $x_D \in \cap \mathcal{E}$. Thus $E$ is closed-complete, and hence $|\mathcal{D}| = |E|$ is Ulam-nonmeasurable [9, 12.2]. Moreover, if $\mathcal{J}$ is a free closed ultrafilter on $X$, then $\{X - F : F \in \mathcal{J} \}$ has a $\theta$-penetration as in the proof of 2.5.

(2) $\Rightarrow$ (1): Suppose there is a countably complete free closed ultrafilter $\mathcal{J}$ on $X$. By (2) and 2.2, $\{X - F : F \in \mathcal{J} \}$ has a weak $\theta$-refinement $\cup_{n<\omega} \mathcal{G}_n'$, and there exists $n < \omega$ such that $A = \{ x \in X : 0 < \text{ord}(x, \mathcal{G}_n) < \omega \}$ meets every member of $\mathcal{J}$. Then $F^* \subseteq \cup \mathcal{G}_n$ for some $F^* \in \mathcal{J}$, and by 2.3 there exists $D \subseteq A \cap F^*$ such that:

(a) if $x, y \in D$ with $x \neq y$, then $x \not\in \text{st}(y, \mathcal{G}_n)$;

(b) $A \cap F^* \subseteq \cup_{x \in D} \text{st}(x, \mathcal{G}_n)$;

(c) $\{ \text{cl}(x) : x \in D \}$ is discrete in $X$.

By (c) and (2), $|D|$ is Ulam-nonmeasurable, and a contradiction follows precisely as in the proof of (b) $\Rightarrow$ (a) of [3, 4.1].
3.2. Remarks. (a) When $X$ is $\mathbb{T}_1$, the cardinality condition of 3.1(1) can be replaced by the requirement that each closed discrete subset of $X$ has Ulam-nonmeasurable cardinality.

(b) The atomicity hypothesis cannot be omitted in the implication (1) $\Rightarrow$ (2) of 3.1: Let $Y$ be the space $(\omega, J)$, where $J = \{\omega\} \cup \{[0, n) : n < \omega\}$, and for $\kappa$ an arbitrary (perhaps Ulam-measurable) cardinal, let $X$ be the topological sum $\sum_{\xi < \kappa} (Y \times [\xi])$. For each $n < \omega$, let $F_n = \sum_{\xi < \kappa} ([n, \omega) \times \{\xi\})$, and note that if $J$ is any closed ultrafilter on $X$, then $F_n \in J$. Since $\bigcap_{n < \omega} F_n = \emptyset$, $X$ is (vacuously) closed-complete.

3.3. Corollary. If $X$ is $\mathbb{T}_1$ and $\theta$-penetrable (in particular, if $X$ has a $\theta$-separating open cover), and if the cardinal of each closed discrete subset of $X$ is Ulam-nonmeasurable, then $X$ is closed-complete.

A space $X$ is cb [16] if for each decreasing sequence $(F_n)_{n < \omega}$ of closed subsets of $X$ with $\bigcap_{n < \omega} F_n = \emptyset$ there is a sequence $(Z_n)_{n < \omega}$ of zero-sets of $X$ with $Z_n \supseteq F_n$ for each $n$ and $\bigcap_{n < \omega} Z_n = \emptyset$. Every Tychonoff closed-complete cb-space is realcompact [7, 1.10], and every normal countably paracompact space is cb [16], so we have 3.4 and 3.5:

3.4. Corollary. If $X$ is a Tychonoff $\theta$-penetrable cb-space such that each closed discrete subset of $X$ has Ulam-nonmeasurable cardinality, then $X$ is realcompact.
3.5. Corollary. If $X$ is a normal countably paracompact $\theta$-penetrable $T_1$-space such that each closed discrete subset of $X$ has Ulam-nonmeasurable cardinality, then $X$ is realcompact.

3.6. Remarks. Corollaries 3.4 and 3.5 generalize Katetov's classical result on realcompactness of paracompact spaces ([15]; cf. [9, 15.20]). (For references to earlier generalizations, see [3].) We note that in 3.4 (resp. 3.5) "cb" (resp. "countably paracompact") cannot be omitted (see the examples in [3, 4.9(d),(e)]).

4. Weakly Separating Covers

We shall say that a cover $\mathcal{P}$ of a space $X$ is weakly separating if for each $x, y \in X$ with $x \neq y$ there is a finite subcollection $\mathcal{A}$ of $\mathcal{P}$ with $x \in \text{int}(\cup \mathcal{A})$ and $y \notin \cup \mathcal{A}$.

4.1. Theorem. Assume $X$ has countable tightness [8, 1.7.13]. If $X$ is $\omega_1$-compact and has a point-countable weakly separating cover, then $X$ is Borel-complete.

4.2. Remarks. Point-countable weakly separating covers are studied in detail in [5] (without being named). Obviously every separating open cover of $X$ is weakly separating, so 4.1 implies 2.8 for spaces of countable tightness. We do not know, however, whether there is an $\omega_1$-compact space of countable tightness with a point-countable weakly separating cover but with no point-countable separating open cover. (If the requirement of countable tightness is omitted, there is such a space.
[4, 4.4], and if that of \( \omega_1 \)-compactness is omitted, there is again such a space (in fact, a locally compact Moore space; see [5, Footnote 4]). On the other hand, if \( X \) has a \( \sigma \)-locally finite separating closed cover \( \mathcal{E} \) (cf. [5, 5.3]), and if \( X \) is \( \omega_1 \)-compact, then \( \mathcal{E} \) is countable and \( \{ X - E : E \in \mathcal{E} \} \) is a countable separating open cover of \( X \). We also do not know whether the hypothesis of countable tightness can be omitted in 4.1.

Before proving 4.1, we systematize and elaborate certain techniques drawn from [5]. Lemma 4.3 generalizes a classical result on open covers [17, Chap. 1, Theorem 18], and 4.5 improves [5, 7.1]. (A more general version of 4.3 (analogous to 2.3) can be proved, but will not be needed here.)

Denote the power set of \( X \) by \( \mathcal{P}(X) \), and if \( E \) is a set, let \( [E]^{<\omega} = \{ F \in \mathcal{P}(E) : |F| < \omega \} \). For \( A \in [\mathcal{P}(X)]^{<\omega} \), set
\[
M(A) = \{ x \in \text{int}(\bigcup A) : x \notin \text{int}(\bigcup B) \text{ if } B \subseteq A, B \neq A \};
\]
and if \( \Phi \subseteq [\mathcal{P}(X)]^{<\omega} \) and \( x \in X \), set
\[
\Phi \langle x \rangle = \{ \text{int}(\bigcup B) : B \subseteq A \text{ for some } A \in \Phi \text{ and } x \in M(B) \}
\]
and
\[
\text{neb}(x, \Phi) = \bigcup \Phi \langle x \rangle.
\]
(We call \( \text{neb}(x, \Phi) \) the nebula of \( x \) with respect to \( \Phi \). Note that if \( U \) is an open collection in \( X \), then \( \text{neb}(x, [U]^{<\omega}) = \text{st}(x, U) \).) The following is easily verified:
4.2. Lemma. (1) If $A \in [\mathcal{P}(X)]^{<\omega}$, then $\text{int}(\cup A) = \cup\{\text{M}(B) : B \subseteq A\}$.

(2) If $\phi \subseteq [\mathcal{P}(X)]^{<\omega}$ and $X = \cup\{\text{int}(\cup A) : A \in \phi\}$, then $x \in \text{neb}(x, \phi)$ for every $x \in X$.

4.3. Lemma. Let $\phi \subseteq [\mathcal{P}(X)]^{<\omega}$ with $X = \cup\{\text{int}(\cup A) : A \in \phi\}$, and let $\prec$ be a well-ordering of $X$. Then there is a subset $D$ of $X$ such that:

(1) If $x, y \in D$ with $x \prec y$, then $y \not\in \text{neb}(x, \phi)$.

(2) $X = \bigcup_{x \in D} \text{neb}(x, \phi)$.

Moreover, if $X$ is $T_1$, then $D$ is closed discrete in $X$.

Proof. By Zorn's lemma, there is a subset $D$ of $X$ maximal with respect to (1) and (2'): if $z \in X$ and $z \prec y$ for some $y \in D$, then $z \in \bigcup_{x \in D} \text{neb}(x, \phi)$. If $D$ fails to satisfy (2) and $u$ is the first element of $X - \bigcup_{x \in D} \text{neb}(x, \phi)$, then $u \not\in D$ (by 4.2(2)) while $D \cup \{u\}$ satisfies (1) and (2'), a contradiction. Thus $D$ satisfies (1) and (2). If $X$ is $T_1$ and if $D$ has a limit point in $X$, then $|D \cap \text{int}(\cup A)| \geq \omega$ for some $A \in \phi$. By 4.2(1), $|D \cap \text{M}(B)| \geq \omega$ for some $B \subseteq A$. Choose $x, y \in D \cap \text{M}(B)$ with $x \prec y$. Then $\text{int}(\cup B) \in \phi(x)$, so $y \in \text{M}(B) \subseteq \text{int}(\cup B) \subseteq \text{neb}(x, \phi)$, a contradiction. Thus $D$ is closed discrete.

4.4. Lemma. Assume $X$ has countable tightness. If $\mathcal{P}$ is a point-countable collection of subsets of $X$, if $\phi \subseteq [\mathcal{P}]^{<\omega}$, and if $x \in X$, then $\phi(x)$ is countable.

Proof. This is an immediate consequence of [5, 2.2].

4.5. Lemma. Let $X$ be an $\omega_1$-compact $T_1$-space with countable tightness. If $\mathcal{P}$ is a point-countable collection
of subsets of $X$ and if $\mathcal{U}$ is a cover of $X$ with $\mathcal{U} = \{\text{int}(\mathcal{u}A) : A \in [\mathcal{P}]^{<\omega}\}$, then $\mathcal{U}$ has a countable subcover.

Proof. Let $\Phi = \{A \in [\mathcal{P}]^{<\omega} : \text{int}(\mathcal{u}A) \subseteq U \text{ for some } U \in \mathcal{U}\}$ and note that $X = \bigcup \{\text{int}(\mathcal{u}A) : A \in \Phi\}$. By 4.3, there is a closed discrete, hence countable, subset $D$ of $X$ such that $X = \bigcup_{x \in D} \text{neb}(x, \Phi)$. Thus, by 4.4, $\bigcup_{x \in D} \Phi(x)$ is a countable refinement of $\mathcal{U}$, and the result follows.

Proof of 4.1. Let $\mathcal{P}$ be a point-countable weakly separating cover of $X$ and let $\mathcal{J}$ be a countably complete Borel ultrafilter on $X$. Clearly $X$ is $T_1$. If for each $x \in X$ there exists $U_x \in [\mathcal{P}]^{<\omega}(x)$ such that $X - U_x \notin \mathcal{J}$, then, by 4.5, the cover $\{U_x : x \in X\}$ of $X$ has a countable subcover; since $\mathcal{J}$ is countably complete, this is a contradiction. Thus there exists $x \in X$ such that $X - U \notin \mathcal{J}$ for all $U \in [\mathcal{P}]^{<\omega}(x)$, and hence for all $U \in [\mathcal{P}]^{<\omega}(x)$ there is $F(U) \in \mathcal{J}$ with $F(U) \subseteq U$. Since $\mathcal{P}$ is weakly separating, and in view of 4.2(1), we have $\cap\{F(U) : U \in [\mathcal{P}]^{<\omega}(x)\} = \cap[\mathcal{P}]^{<\omega}(x) = \{x\}$. But $[\mathcal{P}]^{<\omega}(x)$ is countable by 4.4, and hence $\{x\} \in \mathcal{J}$. Thus $x \in \cap \mathcal{J}$.

References


Ohio University

Athens, Ohio 45701