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1. Introduction

For $\mathcal{S}$ a collection of subsets of a topological space X and $x \in X$, set $\mathcal{S}(x) = \{S \in \mathcal{S}: x \in S\}$, $I(x,\mathcal{S}) = \cap\mathcal{S}(x)$, $st(x,\mathcal{S}) = \cup\mathcal{S}(x)$, and $ord(x,\mathcal{S}) = |\mathcal{S}(x)|$. ($|E|$ denotes the cardinal of the set E. Cardinals are initial ordinals.) The following theorem is due to Chaber:

1.1. *Theorem* (Chaber [6, 3.B]). *Let* $\mathcal{U}$ *be an open cover of a countably compact space* X. *If there exists an open cover* $\cup_{n<\omega}\mathcal{G}_n$ *of* X *such that, for every* $x \in X$, $\cap\{I(x,\mathcal{G}_n): n < \omega$ *and* $0 < ord(x,\mathcal{G}_n) \leq \omega\} \subset U$ *for some* $U \in \mathcal{U}$, *then* $\mathcal{U}$ *contains a finite subcover.*

In this note we first prove a theorem (2.4) that quickly yields 1.1, and then obtain several results closely related to 1.1. Some of the latter generalize the main results of [3]. All of our results have cardinal generalizations, but for simplicity only the countable versions of these more general theorems will be considered here.

2. Closed-Completeness of $\delta\theta$-Penetrable Spaces

To state our results succinctly, we shall say that an open cover $\cup_{n<\omega}\mathcal{G}_n$ of a topological space X is a θ-*penetration* (resp. $\delta\theta$-*penetration*) of a cover $\mathcal{U}$ of X if, for every

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[2]Dedicated to Casper Goffman on his 66th birthday.

$x \in X$, $\cap\{I(x,\mathcal{G}_n): n < \omega$ and $0 < \mathrm{ord}(x,\mathcal{G}_n) < \omega\} \subset U$ for some $U \in \mathcal{U}$ (resp. $\cap\{I(x,\mathcal{G}_n): n < \omega$ and $0 < \mathrm{ord}(x,\mathcal{G}_n) \leq \omega\} \subset U$ for some $U \in \mathcal{U}$), and that X is θ-*penetrable* (resp. $\delta\theta$-*penetrable*) if every open cover of X has a θ-penetration (resp. $\delta\theta$-penetration).

2.1. *Remarks.* (a) A cover $\mathcal{G}$ of X is *separating* if for each $x, y \in X$ with $x \neq y$ there exists $G \in \mathcal{G}$ with $x \in G$ and $y \notin G$; and a cover $\cup_{n<\omega}\mathcal{G}_n$ of X is θ-*separating* [12, 3.1] if for every $x \in X$, $\cap\{I(x,\mathcal{G}_n): n < \omega$ and $0 < \mathrm{ord}(x,\mathcal{G}_n) < \omega\} = \{x\}$. Obviously each point-countable separating open cover of X is a $\delta\theta$-penetration of every cover of X, and each θ-separating open cover of X is a θ-penetration of every cover of X.

(b) A *weak* θ-*refinement* (resp. *weak* $\delta\theta$-*refinement*) of a cover $\mathcal{U}$ of X is an open refinement $\cup_{n<\omega}\mathcal{G}_n$ of $\mathcal{U}$ such that $X = \cup_{n<\omega}\{x \in X: 0 < \mathrm{ord}(x,\mathcal{G}_n) < \omega\}$ (resp. $X = \cup_{n<\omega}\{x \in X: 0 < \mathrm{ord}(x,\mathcal{G}_n) \leq \omega\}$) (see [2] and [18]). It is easily seen that (*) every weak θ-refinement (resp. weak $\delta\theta$-refinement) of $\mathcal{U}$ is a θ-penetration (resp. $\delta\theta$-penetration) of $\mathcal{U}$. The converse of (*), however, is false: Let X be an hereditarily separable non-Lindelöf space obtained by refining the usual topology of $\mathbf{R}$ (see [11] and [14]); it suffices to observe that $\mathbf{R}$ (and hence X) has a θ-separating open cover but that, by [3, 3.16], X is not weakly $\delta\theta$-refinable (i.e. some open cover of X has no weak $\delta\theta$-refinement). But (*) has a partial converse; this is the substance of 2.2 below.

By a *closed ultrafilter* on X we mean a maximal filter in the lattice of closed subsets of X. A closed ultrafilter $\mathcal{F}$ on X is *countably complete* if $\cap\mathcal{A} \in \mathcal{F}$ for every $\mathcal{A} \subset \mathcal{F}$ with $|\mathcal{A}| \leq \omega$, and $\mathcal{F}$ is *fixed* (resp. *free*) if $\cap\mathcal{F} \neq \emptyset$ (resp. $\cap\mathcal{F} = \emptyset$). A space X is *closed-complete* (= a-realcompact [7]) if every countably complete closed ultrafilter on X is fixed. (If "closed" is replaced by "Borel" in the preceding definitions, one obtains the definition of a *Borel-complete* space; see [10] and [3, p. 20]. We note that Borel-completeness implies closed-completeness [10, 1.1].)

2.2. *Lemma. If $\mathcal{F}$ is a countably complete free closed ultrafilter on* X *and if $\mathcal{G} = \cup_{n<\omega}\mathcal{G}_n$ is a θ-penetration (resp. $\delta\theta$-penetration) of $\mathcal{U} = \{X - F: F \in \mathcal{F}\}$, then $\mathcal{G}$ has a subcover that is a weak θ-refinement (resp. weak $\delta\theta$-refinement) of $\mathcal{U}$.*

Proof. We write the proof for the case in which $\mathcal{G}$ is a θ-penetration of $\mathcal{U}$. For each $n < \omega$, let

$$A_n = \{x \in X: \text{ord}(x,\mathcal{G}_n) < \omega \text{ and } X - G \in \mathcal{F} \text{ for some } G \in \mathcal{G}_n(x)\}.$$

If there exists $y \in X - \cup_{n<\omega}A_n$, set

$$K = \{n \in \omega: 0 < \text{ord}(y,\mathcal{G}_n) < \omega\},$$

$$M = \{(n,G): n \in K \text{ and } G \in \mathcal{G}_n(y)\}.$$

Then for each $(n,G) \in M$ we have $\text{ord}(y,\mathcal{G}_n) < \omega$, $G \in \mathcal{G}_n(y)$, and $y \notin A_n$, and thus $X - G \notin \mathcal{F}$; hence $F(n,G) \subset G$ for some $F(n,G) \in \mathcal{F}$. Then $\cap_{n\in K}(\cap_{G\in\mathcal{G}_n(y)}F(n,G)) \in \mathcal{F}$. But

$$\cap_{n\in K}(\cap_{G\in\mathcal{G}_n(y)}F(n,G)) \subset \cap_{n\in K}I(y,\mathcal{G}_n) \subset U$$

for some $U \in \mathcal{U}$, a contradiction, and we conclude that $X = \bigcup_{n<\omega} A_n$. Now for each $n < \omega$ and each $x \in A_n$, there is $G(x,n) \in \mathcal{G}_n(x)$ with $G(x,n) \in \mathcal{U}$. For each $n < \omega$, let $\mathcal{G}_n^* = \{G(x,n): x \in A_n\}$, and let $\mathcal{G}^* = \bigcup_{n<\omega} \mathcal{G}_n^*$. Note that if $x \in X$, then $x \in A_n$ for some n. Then $x \in G(x,n) \in \mathcal{G}_n^*$ and $|\mathcal{G}_n^*(x)| \leq |\mathcal{G}_n(x)| < \omega$, so $\mathcal{G}^*$ is a weak θ-refinement of $\mathcal{U}$.

2.3. *Lemma* (cf. [17, Chap. 1, Theorem 18]). *Let* $A \subset X$ *and let* $\mathcal{G}$ *be a collection of open subsets of* X *such that* $\text{cl } A \subset \bigcup\mathcal{G}$. *Then there exists* $D \subset A$ *such that:*

(1) If $x, y \in D$ *with* $x \neq y$, *then* $x \notin \text{st}(y,\mathcal{G})$.

(2) $A \subset \bigcup_{x \in D} \text{st}(x,\mathcal{G})$.

(3) $\{\text{cl}\{x\}: x \in D\}$ *is discrete in* X.

Proof. By Zorn's lemma, there exists $D \subset A$ maximal with respect to (1), and then D must satisfy (2). If $\mathcal{D} = \{\text{cl}\{x\}: x \in D\}$ is not discrete in X, there is $p \in \text{cl } A$ such that every neighborhood of p meets at least two distinct members of $\mathcal{D}$. Then $p \in G$ for some $G \in \mathcal{G}$, so there exist $x, y \in D$ with $x \neq y$ such that $G \cap \text{cl}\{x\} \neq \emptyset$ and $G \cap \text{cl}\{y\} \neq \emptyset$. But then $x \in \text{st}(y,\mathcal{G})$, contrary to (1).

The *discreteness character* $\Delta(X)$ of a space X is $\omega\cdot\kappa$, where $\kappa = \sup\{|\mathcal{D}| : \mathcal{D}$ is a discrete collection of nonempty closed subsets of X$\}$ [13, §3]. (For a T_1-space X, $\Delta(X)$ is the extent of X [8, 1.7.12] and X is ω_1-compact (i.e. every closed discrete subset of X is countable) if and only if $\Delta(X) = \omega$ [13, 3.2].)

2.4. *Theorem*. *If* $\Delta(X) = \omega$ *and if* $\mathcal{F}$ *is a free closed ultrafilter on* X *such that* $\{X - F: F \in \mathcal{F}\}$ *has a*

$\delta\theta$-penetration, then $\mathcal{F}$ is not countably complete.

Proof. If $\mathcal{F}$ is countably complete, then, by 2.2, $\{X - F: F \in \mathcal{F}\}$ has a weak $\delta\theta$-refinement $\cup_{n<\omega}\mathcal{G}_n$, and there exists $n < \omega$ such that $A = \{x \in X: 0 < \text{ord}(x,\mathcal{G}_n) \leq \omega\}$ meets every member of $\mathcal{F}$. Since $A \subset \cup\mathcal{G}_n$, we have $F^* \subset \cup\mathcal{G}_n$ for some $F^* \in \mathcal{F}$. By 2.3 there exists $D \subset A \cap F^*$ with $A \cap F^* \subset \cup_{x\in D}\text{st}(x,\mathcal{G}_n)$ and $|D| \leq \Delta(X) = \omega$. Then $\mathcal{W} = \cup_{x\in D}\mathcal{G}_n(x)$ is countable, and for each $W \in \mathcal{W}$ there is $F(W) \in \mathcal{F}$ with $W \subset X - F(W)$. But then $A \cap F^* \cap (\cap_{W\in\mathcal{W}}F(W)) = \emptyset$, a contradiction.

We obtain Chaber's theorem as follows:

Proof of 1.1. If the conclusion fails, then $\{X - U: U \in \mathcal{U}\} \subset \mathcal{F}$ for some (free) closed ultrafilter $\mathcal{F}$ on X, and by the hypothesis of 1.1, $\{X - F: F \in \mathcal{F}\}$ has a $\delta\theta$-penetration. But since X is countably compact, $\Delta(X) = \omega$ and $\mathcal{F}$ is countably complete. This contradicts 2.4.

The following generalizes [3, 3.2]:

2.5. *Corollary. If $\Delta(X) = \omega$, then the following are equivalent:*

(1) X *is closed-complete.*

(2) If $\mathcal{F}$ is any free closed ultrafilter on X, *then $\{X - F: F \in \mathcal{F}\}$ has a $\delta\theta$-penetration.*

Proof. If X is closed-complete and $\mathcal{F}$ is a free closed ultrafilter on X, then $\cap_{n<\omega}F_n = \emptyset$ for some sequence $(F_n)_{n<\omega}$ of members of $\mathcal{F}$, and clearly $\cup_{n<\omega}\{X - F_n\}$ is a θ-penetration of $\{X - F: F \in \mathcal{F}\}$. The converse is immediate from 2.4.

A space X is *isocompact* [1] if every countably compact closed subset of X is compact. We shall say that X is *iso-closed-complete* (resp. *iso-Lindelöf*) if every closed subset of X with countable discreteness character is closed-complete (resp. Lindelöf). Clearly every iso-Lindelöf space is iso-closed-complete, and since countably compact closed-complete spaces are compact [3, 3.6], every iso-closed-complete space is isocompact. Since $\delta\theta$-penetrability is closed-hereditary, Chaber's theorem evidently implies that $\delta\theta$-penetrable spaces are isocompact. More generally:

2.6. *Corollary. Every $\delta\theta$-penetrable space is iso-closed-complete.*

2.7. *Remarks.* The example of 2.1(b) shows that hereditarily θ-penetrable regular T_1-spaces need not be iso-Lindelöf. For an example of an isocompact space that is not iso-closed-complete, let X be the subspace of ω_2 obtained by deleting all nonisolated points having a countable base (see [9, 9L]). Then every countably compact closed subset of X is finite (so X is isocompact), and X is ω_1-compact. But X is normal, countably paracompact, and nonrealcompact, and thus not closed-complete [7, 1.10]. (This example was pointed out to the author by Eric van Douwen.)

It follows from 2.5 that an ω_1-compact space with a point-countable separating open cover is closed-complete. But in this case a stronger result is available:

2.8. *Theorem. If* X *is an* ω_1*-compact space with a point-countable separating open cover, then* X *is Borel-complete.*

Proof. Let $\mathcal{U}$ be a point-countable separating open cover of X and let $\mathcal{F}$ be a countably complete Borel ultrafilter on X. Suppose that for each $x \in X$ there exists $U_x \in \mathcal{U}(x)$ with $X - U_x \in \mathcal{F}$. Let $\mathcal{G} = \{U_x: x \in X\}$. By 2.3 there exists $D \subset X$ with $X = \cup_{x \in D} st(x, \mathcal{G})$ and $|D| \leq \Delta(X) = \omega$. But then $\cup_{x \in D} \mathcal{G}(x)$ is a countable cover of X, which contradicts the countable completeness of $\mathcal{F}$. Thus there exists $x \in X$ such that $X - U \notin \mathcal{F}$ for every $U \in \mathcal{U}(x)$, and hence for every $U \in \mathcal{U}(x)$ there is $F(U) \in \mathcal{F}$ with $F(U) \subset U$. Then $\cap\{F(U): U \in \mathcal{U}(x)\} \subset \cap\mathcal{U}(x) = \{x\}$, and since $\mathcal{U}(x)$ is countable, we have $\{x\} \in \mathcal{F}$. Thus $x \in \cap\mathcal{F}$.

3. Closed-Completeness of θ-Penetrable Spaces

The lattice of closed subsets of a space X is *atomic* if each nonempty closed subset of X contains a minimal nonempty closed set. (This holds, for example, if X is essentially T_1, i.e. for each $x, y \in X$, either $cl\{x\} \cap cl\{y\} = \emptyset$ or $cl\{x\} = cl\{y\}$.) The following generalizes [3, 4.1]:

3.1. *Theorem. If the lattice of closed subsets of* X *is atomic, then the following are equivalent:*

(1) X *is closed-complete.*

(2) The cardinal of each discrete collection of closed subsets of X *is Ulam-nonmeasurable, and if* $\mathcal{F}$ *is*

any free closed ultrafilter on X*, then* $\{X - F\colon F \in \mathcal{F}\}$ *has a* θ*-penetration.*

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{D}$ be a discrete collection of nonempty closed subsets of X; we may assume that each $D \in \mathcal{D}$ is minimal. For each $D \in \mathcal{D}$, choose $x_D \in D$, let $E = \{x_D\colon D \in \mathcal{D}\}$, and let $\mathcal{E}$ be a countably complete ultrafilter on the (discrete) space E. Let $\mathcal{E}^* = \{F\colon F$ is closed in X and $F \cap E \in \mathcal{E}\}$. The minimality of the members of $\mathcal{D}$ allows one to conclude that $\mathcal{E}^*$ is a countably complete closed ultrafilter on X, and hence, by (1), there exists $y \in \cap\mathcal{E}^*$. Since $\cup\mathcal{D} \in \mathcal{E}^*$, $y \in D$ for some $D \in \mathcal{D}$, and it follows that $x_D \in \cap\mathcal{E}$. Thus E is closed-complete, and hence $|\mathcal{D}| = |E|$ is Ulam-nonmeasurable [9, 12.2]. Moreover, if $\mathcal{F}$ is a free closed ultrafilter on X, then $\{X - F\colon F \in \mathcal{F}\}$ has a θ-penetration as in the proof of 2.5.

(2) $\Rightarrow$ (1): Suppose there is a countably complete free closed ultrafilter $\mathcal{F}$ on X. By (2) and 2.2, $\{X - F\colon F \in \mathcal{F}\}$ has a weak θ-refinement $\cup_{n<\omega}\mathcal{G}_n$, and there exists $n < \omega$ such that $A = \{x \in X\colon 0 < \mathrm{ord}(x,\mathcal{G}_n) < \omega\}$ meets every member of $\mathcal{F}$. Then $F^* \subset \cup\mathcal{G}_n$ for some $F^* \in \mathcal{F}$, and by 2.3 there exists $D \subset A \cap F^*$ such that:

(a) if $x, y \in D$ with $x \neq y$, then $x \notin \mathrm{st}(y,\mathcal{G}_n)$;

(b) $A \cap F^* \subset \cup_{x\in D}\mathrm{st}(x,\mathcal{G}_n)$;

(c) $\{\mathrm{cl}\{x\}\colon x \in D\}$ is discrete in X.

By (c) and (2), $|D|$ is Ulam-nonmeasurable, and a contradiction follows precisely as in the proof of (b) $\Rightarrow$ (a) of [3, 4.1].

3.2. *Remarks.* (a) When X is T_1, the cardinality condition of 3.1(1) can be replaced by the requirement that each closed discrete subset of X has Ulam-nonmeasurable cardinality.

(b) The atomicity hypothesis cannot be omitted in the implication (1) $\Rightarrow$ (2) of 3.1: Let Y be the space $(\omega, \mathcal{J})$, where $\mathcal{J} = \{\omega\} \cup \{[0,n): n < \omega\}$, and for κ an arbitrary (perhaps Ulam-measurable) cardinal, let X be the topological sum $\Sigma_{\xi<\kappa}(Y \times \{\xi\})$. For each $n < \omega$, let $F_n = \Sigma_{\xi<\kappa}([n,\rightarrow) \times \{\xi\})$, and note that if $\mathcal{F}$ is any closed ultrafilter on X, then $F_n \in \mathcal{F}$. Since $\bigcap_{n<\omega} F_n = \emptyset$, X is (vacuously) closed-complete.

3.3. *Corollary. If* X *is* T_1 *and* θ*-penetrable (in particular, if* X *has a* θ*-separating open cover), and if the cardinal of each closed discrete subset of* X *is Ulam-nonmeasurable, then* X *is closed-complete.*

A space X is cb [16] if for each decreasing sequence $(F_n)_{n<\omega}$ of closed subsets of X with $\bigcap_{n<\omega} F_n = \emptyset$ there is a sequence $(Z_n)_{n<\omega}$ of zero-sets of X with $Z_n \supset F_n$ for each n and $\bigcap_{n<\omega} Z_n = \emptyset$. Every Tychonoff closed-complete cb-space is realcompact [7, 1.10], and every normal countably paracompact space is cb [16], so we have 3.4 and 3.5:

3.4. *Corollary. If* X *is a Tychonoff* θ*-penetrable* cb*-space such that each closed discrete subset of* X *has Ulam-nonmeasurable cardinality, then* X *is realcompact.*

3.5. *Corollary. If* X *is a normal countably paracompact* θ*-penetrable* T_1*-space such that each closed discrete subset of* X *has Ulam-nonmeasurable cardinality, then* X *is realcompact.*

3.6. *Remarks.* Corollaries 3.4 and 3.5 generalize Katětov's classical result on realcompactness of paracompact spaces ([15]; cf. [9, 15.20]). (For references to earlier generalizations, see [3].) We note that in 3.4 (resp. 3.5) "cb" (resp. "countably paracompact") cannot be omitted (see the examples in [3, 4.9(d),(e)]).

4. Weakly Separating Covers

We shall say that a cover $\mathcal{P}$ of a space X is *weakly separating* if for each x, y $\in$ X with x $\neq$ y there is a finite subcollection $\mathcal{A}$ of $\mathcal{P}$ with $x \in \mathrm{int}(\cup\mathcal{A})$ and $y \notin \cup\mathcal{A}$.

4.1. *Theorem. Assume* X *has countable tightness* [8, 1.7.13]. *If* X *is* ω_1*-compact and has a point-countable weakly separating cover, then* X *is Borel-complete.*

4.2. *Remarks.* Point-countable weakly separating covers are studied in detail in [5] (without being named). Obviously every separating open cover of X is weakly separating, so 4.1 implies 2.8 for spaces of countable tightness. We do not know, however, whether there is an ω_1-compact space of countable tightness with a point-countable weakly separating cover but with no point-countable separating open cover. (If the requirement of countable tightness is omitted, there is such a space

[4, 4.4], and if that of ω_1-compactness is omitted, there is again such a space (in fact, a locally compact Moore space; see [5, Footnote 4]). On the other hand, if X has a σ-locally finite separating closed cover $\mathcal{E}$ (cf. [5, 5.3]), and if X is ω_1-compact, then $\mathcal{E}$ is countable and $\{X - E: E \in \mathcal{E}\}$ is a countable separating open cover of X. We also do not know whether the hypothesis of countable tightness can be omitted in 4.1.

Before proving 4.1, we systematize and elaborate certain techniques drawn from [5]. Lemma 4.3 generalizes a classical result on open covers [17, Chap. 1, Theorem 18], and 4.5 improves [5, 7.1]. (A more general version of 4.3 (analogous to 2.3) can be proved, but will not be needed here.)

Denote the power set of X by $\mathcal{P}(X)$, and if E is a set, let $[E]^{<\omega} = \{F \in \mathcal{P}(E): |F| < \omega\}$. For $\mathcal{A} \in [\mathcal{P}(X)]^{<\omega}$, set

$$M(\mathcal{A}) = \{x \in \operatorname{int}(\cup\mathcal{A}): x \notin \operatorname{int}(\cup\mathcal{B}) \text{ if } \mathcal{B} \subset \mathcal{A},\ \mathcal{B} \neq \mathcal{A}\};$$

and if $\Phi \subset [\mathcal{P}(X)]^{<\omega}$ and $x \in X$, set

$$\Phi\langle x\rangle = \{\operatorname{int}(\cup\mathcal{B}): \mathcal{B} \subset \mathcal{A} \text{ for some } \mathcal{A} \in \Phi \text{ and } x \in M(\mathcal{B})\}$$

and

$$\operatorname{neb}(x,\Phi) = \cup\Phi\langle x\rangle.$$

(We call $\operatorname{neb}(x,\Phi)$ the *nebula of* x *with respect to* Φ. Note that if $\mathcal{U}$ is an open collection in X, then $\operatorname{neb}(x,[\mathcal{U}]^{<\omega}) = \operatorname{st}(x,\mathcal{U})$.) The following is easily verified:

4.2. *Lemma.* *(1) If* $\mathcal{A} \in [\mathcal{P}(X)]^{<\omega}$*, then* $int(\cup\mathcal{A}) = \cup\{M(\mathcal{B}): \mathcal{B} \subset \mathcal{A}\}$.

(2) If $\Phi \subset [\mathcal{P}(X)]^{<\omega}$ *and* $X = \cup\{int(\cup\mathcal{A}): \mathcal{A} \in \Phi\}$*, then* $x \in neb(x,\Phi)$ *for every* $x \in X$.

4.3. *Lemma.* *Let* $\Phi \subset [\mathcal{P}(X)]^{<\omega}$ *with* $X = \cup\{int(\cup\mathcal{A}): \mathcal{A} \in \Phi\}$*, and let* $<$ *be a well-ordering of* X*. Then there is a subset* D *of* X *such that:*

(1) If $x, y \in D$ *with* $x < y$*, then* $y \notin neb(x,\Phi)$.

(2) $X = \cup_{x\in D} neb(x,\Phi)$.

Moreover, if X *is* T_1*, then* D *is closed discrete in* X.

Proof. By Zorn's lemma, there is a subset D of X maximal with respect to (1) and (2'): if $z \in X$ and $z < y$ for some $y \in D$, then $z \in \cup_{x\in D} neb(x,\Phi)$. If D fails to satisfy (2) and u is the first element of $X - \cup_{x\in D} neb(x,\Phi)$, then $u \notin D$ (by 4.2(2)) while $D \cup \{u\}$ satisfies (1) and (2'), a contradiction. Thus D satisfies (1) and (2). If X is T_1 and if D has a limit point in X, then $|D \cap int(\cup\mathcal{A})| \geq \omega$ for some $\mathcal{A} \in \Phi$. By 4.2(1), $|D \cap M(\mathcal{B})| \geq \omega$ for some $\mathcal{B} \subset \mathcal{A}$. Choose $x, y \in D \cap M(\mathcal{B})$ with $x < y$. Then $int(\cup\mathcal{B}) \in \Phi\langle x\rangle$, so $y \in M(\mathcal{B}) \subset int(\cup\mathcal{B}) \subset neb(x,\Phi)$, a contradiction. Thus D is closed discrete.

4.4. *Lemma.* *Assume* X *has countable tightness. If* $\mathcal{P}$ *is a point-countable collection of subsets of* X*, if* $\Phi \subset [\mathcal{P}]^{<\omega}$*, and if* $x \in X$*, then* $\Phi\langle x\rangle$ *is countable.*

Proof. This is an immediate consequence of [5, 2.2].

4.5. *Lemma.* *Let* X *be an* ω_1*-compact* T_1*-space with countable tightness. If* $\mathcal{P}$ *is a point-countable collection* *of subsets of* X *and if* $\mathcal{U}$ *is a cover of* X *with* $\mathcal{U} \subset \{\text{int}(\cup\mathcal{A})$: $\mathcal{A} \in [\mathcal{P}]^{<\omega}\}$, *then* $\mathcal{U}$ *has a countable subcover.*

Proof. Let $\Phi = \{\mathcal{A} \in [\mathcal{P}]^{<\omega}$: $\text{int}(\cup\mathcal{A}) \subset U$ for some $U \in \mathcal{U}\}$ and note that $X = \cup\{\text{int}(\cup\mathcal{A}): \mathcal{A} \in \Phi\}$. By 4.3, there is a closed discrete, hence countable, subset D of X such that $X = \cup_{x \in D}\text{neb}(x,\Phi)$. Thus, by 4.4, $\cup_{x \in D}\Phi\langle x\rangle$ is a countable refinement of $\mathcal{U}$, and the result follows.

Proof of 4.1. Let $\mathcal{P}$ be a point-countable weakly separating cover of X and let $\mathcal{J}$ be a countably complete Borel ultrafilter on X. Clearly X is T_1. If for each $x \in X$ there exists $U_x \in [\mathcal{P}]^{<\omega}\langle x\rangle$ such that $X - U_x \in \mathcal{J}$, then, by 4.5, the cover $\{U_x: x \in X\}$ of X has a countable subcover; since $\mathcal{J}$ is countably complete, this is a contradiction. Thus there exists $x \in X$ such that $X - U \notin \mathcal{J}$ for all $U \in [\mathcal{P}]^{<\omega}\langle x\rangle$, and hence for all $U \in [\mathcal{P}]^{<\omega}\langle x\rangle$ there is $F(U) \in \mathcal{J}$ with $F(U) \subset U$. Since $\mathcal{P}$ is weakly separating, and in view of 4.2(1), we have $\cap\{F(U): U \in [\mathcal{P}]^{<\omega}\langle x\rangle\} \subset \cap[\mathcal{P}]^{<\omega}\langle x\rangle = \{x\}$. But $[\mathcal{P}]^{<\omega}\langle x\rangle$ is countable by 4.4, and hence $\{x\} \in \mathcal{J}$. Thus $x \in \cap\mathcal{J}$.

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