TOPOLOGY PROCEEDINGS
Volume 5, 1980
Pages 47–57

http://topology.auburn.edu/tp/

PARALINDELÖF SPACES AND CLOSED MAPPINGS

by

Dennis K. Burke

Topology Proceedings

Web: http://topology.auburn.edu/tp/
Mail: Topology Proceedings
Department of Mathematics & Statistics
Auburn University, Alabama 36849, USA
E-mail: topolog@auburn.edu
ISSN: 0146-4124

PARALINDELÖF SPACES AND CLOSED MAPPINGS

Dennis K. Burke

1. Introduction

A space X is said to be *paralindelöf* if every open cover has a locally countable open refinement. Recent interest in these spaces and spaces with σ-locally countable base has generated results in a number of papers ([B_1], [DGN], [F], [FR], [N]). In particular, special note should be given to the examples, presented by C. Navy in [N], of the first known regular (and normal) paralindelöf spaces which are not paracompact.

In this note we study the closed mapping properties of paralindelöf and related spaces. We show that the paralindelöf property is not preserved under closed maps but is preserved under perfect mappings. Perfect mapping results are also given for σ-paralindelöf spaces and spaces with a σ-locally countable base. The characterization of paralindelöf spaces, using refinements weaker than locally countable open refinements, may be of independent interest.

All regular spaces are assumed to be T_1, and all mappings are continuous and onto. The set of natural numbers is denoted by N.

2. Closed Mappings

In contrast to the situation for paracompact spaces (and many other covering properties) we show here that the paralindelöf property is not preserved under closed mappings. The examples below indicate that given any noncollectionwise normal paralindelöf space there corresponds a simple closed map from a paralindelöf space onto a space which is not paralindelöf. Keep in mind that specific examples are not needed here and that C. Navy has given pertinent examples in [N].

2.1 *Example. If* X *is any regular but nonnormal paralindelöf space and* $\sum_{\alpha<\omega_1} X_\alpha$ *denotes the topological sum of* ω_1 *copies of* X *then there exists a closed map* ϕ *from the paralindelöf space* $\sum_{\alpha<\omega_1} X_\alpha$ *onto a nonparalindelöf space* Y.

Proof. Since X is not normal there exist disjoint closed subsets A and B of X which cannot be separated by open sets. Let $E = \sum_{\alpha<\omega} A_\alpha$ denote the topological sum of the corresponding copies of A in X_α and let Y be the quotient space obtained from $\sum_{\alpha<\omega_1} X_\alpha$ by shrinking E to a point p. The corresponding quotient map $\phi\colon \sum_{\alpha<\omega_1} X_\alpha \to Y$ is obviously a closed map. To see that Y is not paralindelöf consider the open cover $\mathcal{U} = \{W\} \cup \{U_\beta\colon \beta < \omega_1\}$ of Y where $W = \phi(\sum_{\alpha<\omega_1} X_\alpha - \sum_{\alpha<\omega_1} B_\alpha)$ (B_α is the copy of B in X_α) and $U_\beta = \phi(X_\beta - E)$. If $\mathcal{U}$ had a locally countable open refinement then there would exist a locally countable open collection $\{V_\beta\colon \beta < \omega_1\}$ in Y such that $\phi(B_\beta) \subset V_\beta \subset U_\beta$ for each $\beta < \omega_1$. Now, if T is an open neighborhood of p (in Y) which meets only countably many V_β there would be some $\gamma < \omega_1$ such that $V_\gamma \cap T = \emptyset$. Then $\phi^{-1}(V_\gamma)$ and $X_\gamma \cap \phi^{-1}(T)$ are disjoint open sets in X_γ containing B_γ and A_γ respectively. This contradicts the condition placed on A and B.

The image space in Example 2.1 is not regular. The next example shows that the paralindelöf property is not always preserved under a closed map even if the domain and range are both normal.

2.2 *Example. If* X *is a normal* (but noncollectionwise normal) *paralindelöf space there is a closed mapping* ϕ: X $\to$ Y, *onto a space* Y *which is not paralindelöf.*

Proof. Since X is not collectionwise normal there is a discrete collection $\{F_\alpha: \alpha \in \Lambda\}$ of closed subsets of X which cannot be separated by open sets. Let Y be the quotient space obtained from X by identifying each F_α with a point p_α and let ϕ: X $\to$ Y be the corresponding quotient map. The map ϕ is clearly closed and Y cannot be paralindelöf. To see this, recall that regular paralindelöf spaces are collectionwise Hausdorff [T]. The set $\{p_\alpha: \alpha \in \Lambda\}$ is a closed discrete set in Y and a "separation" by open sets in Y would induce a "separation" of $\{F_\alpha: \alpha \in \Lambda\}$ by open sets in X. Hence Y cannot be collectionwise Hausdorff and must not be paralindelöf.

3. Perfect Mappings and Paralindelof Spaces

One of the main results of this section is the following characterization of paralindelöf spaces. The question of perfect images (of paralindelöf spaces) is then easily resolved by Corollary 3.2. The present form of Theorem 3.1 is due to J. Chaber, who made suggestions which simplified and strengthened the author's original form. This improvement in Theorem 3.1 also allows the use of Lindelöf fibers instead of compact fibers in Corollary 3.2. We wish to thank J. Chaber for permission to incorporate these improvements in this section.

3.1 *Theorem. For any space* Y *the following are equivalent:*

(a) Y *is paralindelöf.*

(b) For any open cover $\mathcal{U}$ *of* Y *there is a locally countable refinement* $\mathcal{H}$ *of* $\mathcal{U}$ *such that if* $y \in Y$ *then* $y \in \mathrm{int}(\mathrm{st}(y,\mathcal{H}))$.

Proof. The (a) $\rightarrow$ (b) portion is trivial so assume condition (b) is true and suppose $\mathcal{U}$ is an open cover of Y. Let $\mathcal{H}$ be a locally countable refinement as given in (b) and let $\mathcal{V}$ be an open cover of Y such that if $V \in \mathcal{V}$ then V intersects at most countably many elements of $\mathcal{H}$. Now there is a locally countable refinement $\mathcal{P}$ of $\mathcal{V}$ such that if $y \in Y$ then $y \in \mathrm{int}(\mathrm{st}(y,\mathcal{P}))$. For each $H \in \mathcal{H}$ pick $U(H) \in \mathcal{U}$ such that $H \subset U(H)$ and let

$$G(H) = \mathrm{int}(\mathrm{st}(H,\mathcal{P})) \cap U(H).$$

Since $H \subset \mathrm{int}(\mathrm{st}(H,\mathcal{P}))$ it is clear that $\mathcal{G} = \{G(H) : H \in \mathcal{H}\}$ covers Y and hence is an open refinement of $\mathcal{U}$. To show $\mathcal{G}$ is locally countable let $y \in Y$ and let W be an open neighborhood of y such that W intersects only countably many elements of $\mathcal{P}$. Now since W intersects only countably many elements of $\mathcal{P}$ and each $P \in \mathcal{P}$ intersects only countably many elements of $\mathcal{H}$ it follows that W intersects only countably many elements of $\{st(H,\mathcal{P}): H \in \mathcal{H}\}$. This says the collection $\{st(H,\mathcal{P}): H \in \mathcal{H}\}$ is locally countable; hence $\mathcal{G}$ is locally countable and the theorem is proved.

3.2 *Corollary. If* X *is paralindelöf and* $f: X \to Y$ *is a closed mapping, with* $f^{-1}(y)$ *Lindelöf for each* $y \in Y$, *then* Y *is paralindelöf.*

Proof. If $\mathcal{U}$ is an open cover of Y let $\mathcal{W}$ be a locally countable open cover of X which refines $\{f^{-1}(U): U \in \mathcal{U}\}$. It is easy to see that $\mathcal{H} = \{f(W): W \in \mathcal{W}\}$ satisfies the conditions of Theorem 3.1 so Y is paralindelöf.

In the space ω_1, with the order topology, every open cover has a locally countable closed refinement. This shows that Theorem 3.1 cannot be strengthened by dropping the "$y \in \text{int}(st(y,\mathcal{H}))$" condition in (b). It is reasonable to expect the condition "$y \in \text{int}(\cup\mathcal{J})$ for some countable $\mathcal{J} \subset \mathcal{H}$" would suffice here but this will also fail as indicated by Proposition 3.3 (or by the space ω_1). Recall that a space X is said to be σ-*paralindelöf* [FR] if every open cover of X has a σ-locally countable open refinement. Clearly, any space with a σ-locally countable base is σ-paralindelöf and there do exist such regular spaces which are not paralindelöf ([DGN], [FR], [F]). (In the proposition below, the "regular" condition in the hypothesis may be dropped if the "closed" condition on the refinement $\mathcal{H}$ is dropped.)

3.3 *Proposition.* *If* X *is a regular* σ*-paralindelöf space then every open cover* $\mathcal{U}$ *of* X *has a locally countable closed refinement* $\mathcal{H}$ *such that if* $x \in X$ *there is a countable subcollection* $\mathcal{J} \subset \mathcal{H}$ *such that* $x \in \text{int}(\cup\mathcal{J})$.

Proof. If $\mathcal{U}$ is an open cover of X let $\mathcal{W} = \cup_{n=1}^{\infty}\mathcal{W}_n$ be an open cover of X such that $\{\overline{W}: W \in \mathcal{W}\}$ refines $\mathcal{U}$ and each $\mathcal{W}_n$ is locally countable. For each $n \in N$ let $W_n^* = \cup\{W: W \in \mathcal{W}_n\}$ and

$$\mathcal{H}_n = \{\overline{W} - \bigcup_{k<n} W_k^*: W \in \mathcal{W}_n\}.$$

If $\mathcal{H} = \cup_{n=1}^{\infty}\mathcal{H}_n$ it is easily verified that $\mathcal{H}$ is a locally countable closed refinement of $\mathcal{U}$. To check the remaining condition, let $x \in X$ and suppose n is the smallest positive integer such that $x \in W_0 \in \mathcal{W}_n$ for some W_0. Let V be an open neighborhood of x meeting only countably many elements of $\{\overline{W}: W \in \mathcal{W}_k, k \leq n\}$. Let

$$\mathcal{J} = \{H: H \in \mathcal{H}_k, k \leq n, H \cap V \neq \emptyset\};$$

then $\mathcal{J}$ is a countable subcollection of $\mathcal{H}$ and $W_0 \cap V \subset \cup\mathcal{J}$ so $x \in \text{int}(\cup\mathcal{J})$.

As might be expected, after Theorem 3.1, there is a similar characterization of σ-paralindelöf spaces. The proof is very similar to the proof of Theorem 3.1 and is omitted.

3.4 *Theorem.* *For any space* Y *the following are equivalent:*

(a) Y *is* σ*-paralindelöf.*

(b) Any open cover $\mathcal{U}$ *of* Y *has a refinement* $\mathcal{H} = \cup_{n=1}^{\infty}\mathcal{H}_n$

where each $\mathcal{H}_n$ *is locally countable and if* $y \in Y$ *there is some* n *such that* $y \in \text{int}(\text{st}(y,\mathcal{H}_n))$.

3.5 *Corollary. If* X *is a* σ*-paralindelöf space and* $f: X \to Y$ *is a perfect mapping then* Y *is* σ*-paralindelöf.*

Proof. Suppose $\mathcal{U}$ is an open cover of Y and $\mathcal{W} = \cup_{n=1}^{\infty}\mathcal{W}_n$ is an open refinement of $\mathcal{U}$ where each $\mathcal{W}_n$ is locally countable. Without loss of generality we may assume that if $y \in Y$ then $f^{-1}(y) \subset \cup\mathcal{W}_k$, for some k. It follows that $\mathcal{H} = \{f(W): W \in \mathcal{W}\}$ satisfies the conditions in Theorem 3.4 (b) so the corollary is proved.

It is interesting that, in contrast to Corollary 3.2, the compactness condition on the fibers of f in Corollary 3.5 cannot be weakened to a Lindelöf condition. This is illustrated by the following example.

3.6 *Example. There is an example of a regular* σ*-paralindelöf space* X*, a space* Y *which is not* σ*-paralindelöf, and a closed mapping* $f: X \to Y$ *with* $f^{-1}(y)$ *Lindelöf for each* $y \in Y$.

Proof. The space X of Example 2.5 in [FR] is a Moore space with a σ-locally countable base and hence is σ-paralindelöf. With minor changes in notation, the following is a quick review of this space.

$$X = \omega_1 \cup \{(\alpha,\beta,n): \alpha \leq \beta < \omega_1,\ n < \omega\}$$

The points in $\{(\alpha,\beta,n): \alpha < \beta < \omega_1,\ n < \omega\}$ are isolated. For $\beta < \omega_1$ and $n < \omega$ the neighborhoods of the point (β,β,n) are those subsets of X which contain (β,β,n) and all but finitely many elements of $\{(\alpha,\beta,n): \alpha < \beta\}$. For $\alpha < \omega_1$ the neighborhoods of α are those subsets of X which, for some $k < \omega$, contain $\{\alpha\} \cup \{(\alpha,\beta,n): \alpha < \beta < \omega_1, k < n < \omega\}$.

The desired space Y is obtained as a quotient space of X by identifying the set $\{\beta\} \cup \{(\beta,\beta,n): n < \omega\}$ as a single point for each $\beta < \omega_1$. If $f: X \to Y$ is the corresponding quotient map it is clear that each $f^{-1}(y)$ is countable and hence Lindelöf. Since the set $D = \omega_1 \cup \{(\beta,\beta,n): \beta < \omega_1, n < \omega\}$ is closed and discrete, and $f^{-1}(y) \subset D$ whenever $|f^{-1}(y)| > 1$ it follows that f is a closed mapping. We leave to the reader the details of showing that Y is not σ-paralindelöf.

4. Perfect Mappings and Spaces With a σ-Locally Countable Base

We conclude this paper by showing that the σ-locally countable base condition is preserved under a perfect mapping, a result announced in [B_2]. Following the lead of Theorem 3.1, a characterization is given for regular spaces with a σ-locally countable base and the mapping result will follow easily.

4.1 *Theorem. For a regular space* Y *the following are equivalent:*

(a) Y *has a σ-locally countable base.*

(b) Y *has a σ-locally countable cover* $\mathcal{P}$ *such that if* $y \in U \subset Y$, *with* U *open in* Y, *there is a finite subcollection* $\mathcal{F} \subset \mathcal{P}$ *such that* $y \in \mathrm{int}(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$.

Proof. To show the nontrivial direction, suppose

$\mathcal{P} = \cup_{n=1}^{\infty}\mathcal{P}_n$ is the cover as given in (b) where each $\mathcal{P}_n$ is locally countable. Since Y is regular we may assume all elements of $\mathcal{P}$ are closed. For each $n,k \in N$ let

$$\Phi_{nk} = \{\mathcal{F}: \mathcal{F} \subset \cup_{i=1}^{k}\mathcal{P}_i, |\mathcal{F}| = n\}$$

and let

$$\Phi = \cup_{n,k}\Phi_{nk}.$$

For $\mathcal{F} \in \Phi$ and $k \in N$ define

$$\mathcal{M}_k(\mathcal{F}) = \{A \in \cup_{i=1}^{k}\mathcal{P}_i: A \subset \mathrm{int}(\cup\mathcal{F}) \text{ and } A \not\subset \mathrm{int}(\cup\mathcal{R}) \text{ if } \mathcal{R} \subset \mathcal{F}, \mathcal{R} \neq \mathcal{F}\}$$

and

$$\mathcal{B} = \{\mathrm{int}(\cup\mathcal{M}_k(\mathcal{F})): \mathcal{F} \in \Phi_{nk}, n,k \in N\}.$$

To see that $\mathcal{B}$ is a base let $y \in U$, with U open in Y. Pick a minimal $\mathcal{F} \in \Phi$ such that $y \in \mathrm{int}(\mathcal{F}) \subset U$ (so $y \notin \mathrm{int}(\mathcal{R})$ if $\mathcal{R} \subset \mathcal{F}$, $\mathcal{R} \neq \mathcal{F}$). By (b) again there is a minimal $\mathcal{D} \subset \cup_{i=1}^{k}\mathcal{P}_i$ such that

$$y \in \mathrm{int}(\cup\mathcal{D}) \subset \cup\mathcal{D} \subset \mathrm{int}(\cup\mathcal{F}).$$

(Pick k large enough so that $\mathcal{F} \in \Phi_{nk}$, $n = |\mathcal{F}|$). Since the elements of $\mathcal{D}$ are closed and $\mathcal{D}$ is minimal with respect to the property that $y \in \mathrm{int}(\cup\mathcal{D})$ it follows that $y \in \cap\mathcal{D}$. Hence, if $D \in \mathcal{D}$ then $y \in D \subset \mathrm{int}(\cup\mathcal{F})$ and $D \not\subset \mathrm{int}(\cup\mathcal{R})$ if $\mathcal{R} \subset \mathcal{F}$, $\mathcal{R} \neq \mathcal{F}$. This says $\mathcal{D} \subset \mathcal{M}_k(\mathcal{F})$ so

$$y \in \mathrm{int}(\cup\mathcal{D}) \subset \mathrm{int}(\cup\mathcal{M}_k(\mathcal{F})) \subset \mathrm{int}(\cup\mathcal{F}) \subset U.$$

In particular $y \in \mathrm{int}(\cup\mathcal{M}_k(\mathcal{F})) \subset U$ and $\mathrm{int}(\cup\mathcal{M}_k(\mathcal{F})) \in \mathcal{B}$.

To show that $\mathcal{B}$ is σ-locally countable, it suffices to show that

$$\mathcal{B}_{nk} = \{\cup\mathcal{M}_k(\mathcal{F}): \mathcal{F} \in \Phi_{nk}\}$$

is locally countable for every $n,k \in N$. If $y \in Y$ there is a neighborhood V of y such that V intersects only countably many elements of $\cup_{i=1}^{k}\mathcal{P}_i$. Now if $V \cap (\cup\mathcal{M}_k(\mathcal{F})) \neq \emptyset$ for some $\mathcal{F} \in \Phi_{nk}$ then $V \cap A \neq \emptyset$ for some $A \in \mathcal{M}_k(\mathcal{F}) \subset \cup_{i=1}^{k}\mathcal{P}_i$ and $V \cap A \neq \emptyset$ for only countably many $A \in \cup_{i=1}^{k}\mathcal{P}_i$. So it suffices to have the following:

4.2 *Lemma. If* $A \in \cup_{i=1}^{k}\mathcal{P}_i$ *and* $n \in N$ *then* $A \in \mathcal{M}_k(\mathcal{F})$ *for only countably many* $\mathcal{F} \in \Phi_{nk}$.

The proof can be deduced from results in [BM_1, Remark 4.1] and [BM_2, Lemma 2.2]. That concludes the proof of the theorem.

4.3 *Corollary. If* X *is a regular space with a* σ*-locally countable base and* $f\colon X \to Y$ *is a perfect mapping then* Y *has a* σ*-locally countable base.*

References

[A] C. E. Aull, *A survey paper on some base axioms*, Top. Proc. 3 (1978), 1-36.

[B_1] D. K. Burke, *Refinements of locally countable collections*, Top. Proc. 4 (1979), 19-27.

[B_2] ________, *Closed mappings*, Surveys in General Topology, Academic Press, 1980, 1-32.

[BM_1] ________ and E. A. Michael, *On a theorem of V. V. Filippov*, Israel J. Math. 11 (1972), 394-397.

[BM_2] ________, *On certain point-countable covers*, Pacific J. Math. 64 (1976), 79-92.

[DGN] S. W. Davis, G. Gruenhage, and P. J. Nyikos, G_δ*-sets in symmetrizable and related spaces*, Gen. Top. Appl. 9 (1978), 253-261.

[F] W. G. Fleissner, *A collectionwise Hausdorff nonnormal Moore space with a* σ*-locally countable base*, Top. Proc. 4 (1979), 83-97.

[FR] ________ and G. M. Reed, *Paralindelöf spaces and spaces with a σ-locally countable base*, Top. Proc. 2 (1977), 89-110.

[N] C. Navy, *Does regular + paralindelöf imply paracompact?*, UAB Topology Conference, 1980.

[T] F. D. Tall, *Set-theoretic consistency results and topological theorems*, Dis. Math. 148 (1977), 1-53.

Miami University

Oxford, Ohio 45056