TOPOLOGICAL GAMES AND ANALYTIC SETS, II

by

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This contribution is a continuation of the second author's note [3]. In contrast to [3], here the strategic situation of Player II in the game $G(X,Y)$ is considered (definitions below). Assuming $Y$ is a separable metric space, the following is shown: (a) $\Rightarrow$ (b) $\Rightarrow$ (c), where

(a) $Y - X$ contains an analytic set which is not Borel separated from $X$,

(b) Player II has a winning strategy in $G(X,Y)$, and

(c) $Y - X$ contains a copy of the Cantor discontinuum.

Finally, some corollaries are derived and some open questions stated.

We recall the definition of the game $G(X,Y)$ of [3]. Let $X$ be a subset of a topological space $Y$. Player I chooses a sequence $E_1 = \langle E(1,1), E(1,2), \cdots \rangle$ of subsets of $X$ so that $\cup_{E_1} = X$. Then Player II chooses $k_1 \in \mathbb{N}$. Assume inductively that $E_1, k_1, \cdots, E_n, k_n$ have been chosen. Then Player I chooses a sequence $E_{n+1} = \langle E(n+1,1), E(n+1,2), \cdots \rangle$ of subsets of $X$ so that $\cup_{E_{n+1}} = E(n, k_n)$. After this Player II chooses $k_{n+1} \in \mathbb{N}$. Player I wins the play $\langle E_1, k_1, E_2, k_2, \cdots \rangle$ of $G(X,Y)$ if $\cap \{E(n, k_n) : n \in \mathbb{N}\} \subseteq X$, otherwise Player II wins.

A subset $X$ of a topological space $Y$ is said to be a Souslin set in $Y$ (more precisely: a Souslin-F set in $Y$) if there is an indexed family
\{F(k_1,\cdots,k_n): \langle k_1,\cdots,k_n \rangle \in \mathbb{N}^n, n \in \mathbb{N}\}

of closed subsets of Y so that
\[ X = \bigcup \bigcap \{F(k_1,\cdots,k_n): n \in \mathbb{N}\}: \langle k_1,k_2,\cdots \rangle \in \mathbb{N}^n \}.

**Theorem 1** [3]. *Player I has a winning strategy in* \( G(X,Y) \) *iff* \( X \) *is a Souslin set in* \( Y \).

A subset \( X \) of a separable metric space \( Y \) is said to be analytic if either \( X = \emptyset \) or there is a continuous map from the space \( \mathbb{N}^\mathbb{N} \) onto \( X \).

Since Souslin and analytic subsets of Polish spaces coincide ([1], p. 482), we have from Theorem 1

**Corollary 1.** Let \( X \) be a subset of a Polish space \( Y \). *Player I has a winning strategy in* \( G(X,Y) \) *iff* \( X \) *is analytic.*

Two subsets \( X \) and \( Z \) of a separable metric space \( Y \) are said to be Borel separated if there is a Borel set \( B \) in \( Y \) such that \( X \subseteq B \) and \( Z \subseteq Y - B \).

**Theorem 2.** *If* \( X \) *is a subset of a separable metric space* \( Y \) *and* \( Y - X \) *contains an analytic set* \( Z \) *which is not Borel separated from* \( X \), *then Player II has a winning strategy in* \( G(X,Y) \). *In particular, if* \( Y - X \) *is analytic non-Borel, then Player II has a winning strategy in* \( G(X,Y) \).

By Corollary 1 and Theorem 2 we obtain

**Corollary 2.** *If* \( X \) *is analytic or co-analytic in a Polish space* \( Y \), *then the game* \( G(X,Y) \) *is determined.*

**Question 1.** Let \( X \) belong to the \( \sigma \)-algebra generated
generated by analytic subsets of an uncountable Polish space Y. Is then $G(X,Y)$ determined?

**Theorem 3.** If X is a subset of a separable metric space Y and Player II has a winning strategy in $G(X,Y)$, then $Y - X$ contains a copy of the Cantor discontinuum.

**Question 2.** Let X be a Lusin set on the real line R (i.e., X is uncountable and $X \cap F$ is at most countable whenever F is nowhere dense in R). Does Player II have a winning strategy in $G(X,R)$?

A subset X of an uncountable Polish space Y is said to be a Bernstein set if neither X nor $Y - X$ contains a copy of the Cantor discontinuum. Each uncountable Polish space contains a Bernstein set ([1], p. 514). Hence by Corollary 1 and Theorem 3 we obtain

**Corollary 3.** If X is a Bernstein set in an uncountable Polish space Y, then the game $G(X,Y)$ is undetermined.

**Major question:** Is the sufficient condition for the existence of a winning strategy for Player II given in Theorem 2 also necessary? Indeed, granted a winning strategy t for Player II, we are unable to determine the descriptive character of the set of points arising as outcomes of arbitrary plays in $G(X,Y)$ with II playing according to t.

**Proofs.** The proof of Theorem 1 given below differs from that of [3] and, moreover, is much simpler.
Proof of Theorem 1. Let $s$ be a strategy of Player I in $G(X,Y)$. Then $s$ determines a Souslin set $X_s = \bigcup\{\bigcap\{E_s(k_1,...,k_n): n \in \mathbb{N}\}: \langle k_1,k_2,... \rangle \in \mathbb{N}^n \}$ in $Y$ as follows: $E_s(k_1) = s(\emptyset)(k_1)$, $E_s(k_1,k_2) = s(k_1)(k_2)$, $E_s(k_1,k_2,k_3) = s(k_1,k_2)(k_3)$, and so on. It is easy to check that $X_s \supset X$. Clearly, $s$ is a winning strategy iff $X_s = X$. Hence, in particular, if $s$ is a winning strategy of Player I, then $X$ is a Souslin set in $Y$. To prove the converse implication, assume that $X$ is a Souslin set in $Y$, i.e.,

$$X = \bigcup\{\bigcap\{F(k_1,...,k_n): n \in \mathbb{N}\}: \langle k_1,k_2,... \rangle \in \mathbb{N}^n \},$$

where each $F(k_1,...,k_n)$ is closed in $Y$. Let us put

$$E(k_1,...,k_n) = \bigcup\{\bigcap\{F(j_1,...,j_m): m \in \mathbb{N}\}: \langle j_1,j_2,... \rangle \in B(k_1,...,k_n) \},$$

where

$$B(k_1,...,k_n) = \{ \langle i_1,i_2,... \rangle \in \mathbb{N}^n: \langle i_1,...,i_n \rangle = \langle k_1,...,k_n \rangle \}.$$ 

It is easy to verify that for any $\langle k_1,k_2,... \rangle \in \mathbb{N}^n$

$$E(k_1,...,k_n) \subset F(k_1,...,k_n),$$

$\bigcap\{F(k_1,...,k_n): n \in \mathbb{N}\} = \bigcap\{E(k_1,...,k_n): n \in \mathbb{N}\},$

$\bigcup\{E(k): k \in \mathbb{N}\} = X$, and

$\bigcup\{E(k_1,...,k_n,k): k \in \mathbb{N}\} = E(k_1,...,k_n).$

Hence

$$X = \bigcup\{\bigcap\{E(k_1,...,k_n): n \in \mathbb{N}\}: \langle k_1,k_2,... \rangle \in \mathbb{N}^n \}$$

and a winning strategy for Player I can be defined as follows: $s(\emptyset)(k_1) = E(k_1)$, $s(k_1)(k_2) = E(k_1,k_2)$, $s(k_1,k_2)(k_3) = E(k_1,k_2,k_3)$, and so on. The proof of Theorem 1 is complete.
Lemma. Let $X$ and $Z$ be subsets of a topological space $Y$, and let $X = \bigcup \{ X_m : m \in \mathbb{N} \}$ and $Z = \bigcup \{ Z_n : n \in \mathbb{N} \}$. If $X$ and $Z$ are not Borel separated, then there are $m \in \mathbb{N}$ and $n \in \mathbb{N}$ so that $X_m$ and $Z_n$ are not Borel separated.

The lemma is classical, see [1], p. 485 or [2], p. 228, for proof (which is easy).

Proof of Theorem 2. Let us assume that $Y - X$ contains an analytic set $Z$ which is not Borel separated from $X$. Let $f$ be a continuous map from $\mathbb{N}^\mathbb{N}$ onto $Z$ and let $F(j_1, \ldots, j_n) = f(B(j_1, \ldots, j_n))$, where, as before, $B(j_1, \ldots, j_n) = \{ \langle i_1, i_2, \ldots \rangle \in \mathbb{N}^\mathbb{N} : \langle i_1, \ldots, i_n \rangle = \langle j_1, \ldots, j_n \rangle \}$.

Then

$\bigcup \{ F(j) : j \in \mathbb{N} \} = Z,$

$\bigcup \{ F(j_1, \ldots, j_n, j) : j \in \mathbb{N} \} = F(j_1, \ldots, j_n)$, and

$\text{diam } F(j_1, \ldots, j_n) \to 0$ as $n \to \infty$

for each $\langle j_1, j_2, \ldots \rangle \in \mathbb{N}^\mathbb{N}$. We shall define a winning strategy $t$ for Player II in $G(X,Y)$ as follows. Let $E_1 = \langle E(1,1), E(1,2), \ldots \rangle$, where $\cup E_1 = X$. Since $X$ and $Z$ are not Borel separated, it follows from the lemma that there is $k_1 \in \mathbb{N}$ and $j_1 \in \mathbb{N}$ so that $E(1,k_1)$ and $F(j_1)$ are not Borel separated. We set $t(E_1) = k_1$. Let $E_2 = \langle E(2,1), E(2,2), \ldots \rangle$, where $\cup E_2 = E(1,k_1)$. Again by the lemma we infer the existence of $k_2 \in \mathbb{N}$ and $j_2 \in \mathbb{N}$ such that $E(2,k_2)$ and $F(j_1, j_2)$ are not Borel separated. We set $t(E_1, E_2) = k_2$, and so on. Since $E(n,k_n)$ and $F(j_1, \ldots, j_n)$ are not Borel
separated, it follows that
\[ E(n,k_n) \cap F(j_1, \ldots, j_n) \neq 0. \]
Since \( n \in \mathbb{N} \) \( \cap \{F(j_1, \ldots, j_n) : n \in \mathbb{N} \} = \{z\} \subset \mathbb{Z} \), where \( z = f(j_1, j_2, \ldots) \), and \( \text{diam } F(j_1, \ldots, j_n) + 0 \) as \( n \to \infty \), we also have \( z \in n \{E(n,k_n) : n \in \mathbb{N} \} \). Indeed, if \( U \) is an open neighbourhood of \( z \) in \( Y \) and \( n \in \mathbb{N} \), then there is \( m > n \) such that \( F(j_1, \ldots, j_m) \subset U \). Since \( F(j_1, \ldots, j_m) \cap E(m,k_m) \neq 0 \), we have \( E(m,k_m) \cap U \neq 0 \) and so \( E(n,k_n) \cap U \neq 0 \), because \( E(n,k_n) = E(m,k_m) \). Thus \( z \in \bigcap_{n \in \mathbb{N}} E(n,k_n) \). Finally, \( (Y - X) \cap \bigcap_{n \in \mathbb{N}} E(n,k_n) \neq 0 \) and thus \( t \) is a winning strategy for Player II. The proof is complete.

**Proof of Theorem 3.** Let \( t \) be a fixed winning strategy for Player II in \( G(X,Y) \), where \( Y \) is a separable metric space. If \( \langle E_1, k_1, \ldots, E_n, k_n \rangle \) is a partial \( t \)-play (i.e., a partial play of \( G(X,Y) \) in which Player II follows the strategy \( t \)), let \( T(E_1, \ldots, E_n) = E_n(k_n) \), the set determined by Player II's \( n \)th move; let \( T(E_1, \ldots, E_n) = X \) if \( n = 0 \).

Let \( M(E_1, \ldots, E_n) \) be the set of all sequences \( E = \langle E(k) : k \in \mathbb{N} \rangle \) such that \( uE = T(E_1, \ldots, E_n) \); i.e., \( M(E_1, \ldots, E_n) \) is the set of all legal moves by Player I following \( \langle E_1, k_1, \ldots, E_n, k_n \rangle \). Clearly, it will suffice to establish the following:

**Claim 1.** We can associate with every finite sequence \( \langle b_1, \ldots, b_n \rangle \) of 0's and 1's a sequence \( E_{b_1}, \ldots, E_{b_n} \) of subsets of \( X \), so that the following conditions are satisfied:

1. \( E_{b_1}, \ldots, E_{b_n} \in M(E_{b_1}, E_{b_2}, \ldots, \), \( E_{b_2}, \ldots, E_{b_{n-1}} \);
(2) \( T(E_{b_1}, \ldots, E_{b_n}) \) has diameter \(< 1/n;\)

(3) the sets \( T(E_{b_1}, \ldots, E_{b_n}) \) have disjoint closures.

In fact, it will suffice to prove:

Claim 2. Let \( \langle E_1, k_1, \ldots, E_n, k_n \rangle \) be a partial \( t \)-play, and let \( J = \{ T(E_1, \ldots, E_n, E) : E \in M(E_1, \ldots, E_n) \} \). Then there are sets \( F' \), \( F'' \in J \) such that \( F' \) and \( F'' \) have diameter \(< 1/(n+1) \) and \( F' \cap F'' = \emptyset. \)

Proof of Claim 2. Let \( W = \{ y \in Y : \text{every neighbourhood of } y \text{ contains a member of } J \} \). It will suffice to show that \( W \) contains at least two points. Let \( \mathcal{U} = \{ U \subseteq Y : U \text{ is open in } Y, \text{ and } U \text{ contains no member of } J \} \). Then \( \bigcup \mathcal{U} = Y - W. \) Since \( Y \) is a separable metric space, we can write \( Y - W = \bigcup \{ U_n : n \in \mathbb{N}, U_n \in \mathcal{U} \}. \) Now let \( E = T(E_1, \ldots, E_n) \), and let \( E_{n+1} = \{ E \cap W, E \cap U_1, E \cap U_2, \ldots : E \in M(E_1, \ldots, E_n) \}. \) Since \( E \cap U_n \not\in J \), we must have \( T(E_1, \ldots, E_{n+1}) = E \cap W. \) Now \( E \cap W \subseteq X, \) but \( E \cap W \not\subseteq X \) since \( t \) is a winning strategy for Player II in \( G(X,Y) \); hence \( E \cap W \) is infinite. The proof is complete.

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References


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