IRREDUCIBLE SPACES AND PROPERTY

\( b_1 \)

by

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1. Introduction

In an unpublished paper [8] J. Chaber introduced a topological property which he called property $b_1$. Chaber showed that this property plays an important role in the study of metacompact and $\theta$-refinable spaces. Since these classes of spaces are irreducible, it is natural to investigate the relationship between property $b_1$ and irreducibility. A topological space $X$ is irreducible if every open cover of $X$ has an open refinement which is a minimal cover of $X$. Studies of irreducible spaces have been made by R. Arens and J. Dugundji [1], J. Boone [3,4], U. Christian [9,10], the author [17,18,19], and J. Worrell and H. Wicke [21].

In this paper we investigate property $b_1$ and its natural variations. In particular we show in Section 2 that property $b_1$ is actually stronger than the notion of weakly $\theta$-refinable but a weaker version of property $b_1$ is implied by weakly $\theta$-refinable. Also in Section 3 we show that another weaker version of property $b_1$ always implies irreducibility. Application of these results are given in Section 4 where several unanswered questions are solved. A number of new problems are also included.

The following notions and definitions are included for the benefit of the reader.
Notation. Let \( J = \{ F_\alpha : \alpha \in A \} \) be a collection of subsets of a space \( X \). We will denote \( \bigcup F_\alpha \) by \( \cup_j \).

**Definition 1.1.** A space \( X \) is called weakly \( \theta \)-refinable provided every open cover \( \mathcal{G} \) of \( X \) has a refinement \( \cup_{i=1}^\infty \mathcal{G}_i \) satisfying:

(i) each \( \mathcal{G}_i = \{ G(\alpha, i) : \alpha \in A_1 \} \) is a collection of open subsets of \( X \),

(ii) for each \( x \in X \), there exists an integer \( n(x) \) such that \( 0 < \text{ord}(x, \mathcal{G}_n(x)) < \infty \),

(iii) if \( x \in X \), then \( x \in G^*_i \) for only finitely many \( i \), where \( G^*_i = \bigcup \mathcal{G}_i \).

Naturally, a cover \( \cup_{i=1}^\infty \mathcal{G}_i \) satisfying (i)-(iii) above is called a weak \( \theta \)-cover. Spaces satisfying only (i) and (ii) are called weakly \( \theta \)-refinable and were introduced by Bennett and Lutzer [2].

**Definition 1.2.** A space \( X \) is called \( \theta \)-refinable if every open cover \( \mathcal{G} \) of \( X \) has a refinement \( \cup_{i=1}^\infty \mathcal{G}_i \) where each \( \mathcal{G}_i \) is an open cover of \( X \) and property (ii) above is satisfied.

The following property was introduced by J. Chaber in an unpublished paper [8]. This property was shown to play an important role in the study of \( \theta \)-refinable and metacompact spaces as stated in the next theorem.

**Definition 1.3.** A space \( X \) is said to have property \( b_1 \) if each open cover \( \mathcal{U} \) of \( X \) can be refined by a cover \( J = \cup_{i=1}^\infty J_i \) such that,
\( J \) is a locally finite collection of closed sets in \( X - \bigcup_{k<n} [uJ_k] \).

**Theorem 1.4.** (1) A space \( X \) is metacompact iff \( X \) is almost expandable and has property \( b_1 \).

(2) A space \( X \) is \( \theta \)-refinable iff \( X \) is almost \( \theta \)-expandable and has property \( b_1 \).

Properties of almost expandable and almost \( \theta \)-expandable spaces are discussed in [8,13,14,16,17,20].

**Definition 1.5.** A collection \( J = \{F_a : a \in A\} \) is called hereditarily closure-preserving (HCP) provided for every \( B \subseteq A \) and every collection \( \{H_\beta : \beta \in B\} \), where \( H_\beta \subseteq F_\beta \), we have that \( \bigcup_{\beta \in B} H_\beta = \bigcup_{\beta \in B} H_\beta \).

**Definition 1.6.** A space \( X \) is said to have property \( B(D(\text{resp.} LF, HCP),a) \) if each open cover \( U \) of \( X \) has a refinement \( \bigcup J_s \), such that for each \( s < a \)

\begin{enumerate}
    \item \( J_s \) is a discrete (resp. locally finite, HCP) collection of closed sets in \( X - \bigcup_{s' < s} [uJ_{s'}]\).
    \item \( \bigcup_{s' < s} [uJ_{s'}] \) is closed in \( X \).
\end{enumerate}

**Remark.** Note that property \( B(LF, \omega_0) \equiv \text{property } b_1 \) according to Chaber [8]. It should be clear that property \( B(D,a) \Rightarrow \text{property } B(LF,a) \Rightarrow \text{property } B(HCP,a) \) for each \( a \).

**Definition 1.7.** A collection \( V \) is a "partial" refinement of a collection \( U \) provided each member of \( V \) is contained in some member of \( U \). (It need not be the case that \( UV = UU \).)
2. Property B(D, ω₀) and Weakly $\overline{\theta}$-Refinable Spaces

In order to begin our study it is interesting to note that property B(D, ω₀) is stronger than the property of weak $\overline{\theta}$-refinability.

**Theorem 2.1.** If a space $X$ has property B(D, ω₀) then $X$ is weakly $\overline{\theta}$-refinable.

**Proof.** Let $\mathcal{U}$ be an open cover of $X$. Then $\mathcal{U}$ has a refinement $\mathcal{U} \supseteq \mathcal{J}_i$ satisfying (1) and (2) in Definition 1.6 above. We now construct the sequence $\{\mathcal{G}_i\}_{i=1}^\infty$ satisfying properties (i)-(iii) of Definition 1.1 above.

Now for each $a \in A$ and each $n < \omega_0$, choose $U(a,n) \in \mathcal{J}_n$. Define $G(a,n) = U(a,n) - \bigcup F(a,n) - \bigcup_{k<n} [U_J^k]$ for each $a \in A$ and $n < \omega_0$ and let $\mathcal{G}_n = \{G(a,n): a \in A\}$. It is clear that each $\mathcal{G}_n$ is a collection of open subsets of $X$. Furthermore if $x \in X$ choose $n(x)$ to be the first integer for which $x$ belongs to some member $F(a,n(x))$ of $\mathcal{J}_n(x)$. Then $x$ belongs to only $G(a,n(x)) \in \mathcal{G}_n(x)$ and $x$ belongs to no member of $\mathcal{G}_k$ for $k > n(x)$. Therefore $\bigcup_{i=1}^\infty \mathcal{G}_i$ satisfies properties (i)-(iii) in Definition 1.1 above so that $X$ is weakly $\overline{\theta}$-refinable.

**Remark.** The author conjectures that property B(D, ω₀) and weakly $\overline{\theta}$-refinability are not equivalent. In fact, the author conjectures that there is a space $X$ which is weakly $\overline{\theta}$-refinable and has property B(D, ω₀+1) but does not
have property $B(D,\omega_0)$. Such examples however appear to be somewhat complicated.

**Theorem 2.2.** Every weakly $\theta$-refinable space has property $B(D,(\omega_0)^2)$.

**Proof.** Let $\bigcup_{i=1}^{\omega}\mathcal{G}_i$ be a weak $\theta$-cover of $X$ where $\mathcal{G}_i = \{G(\alpha,i) : \alpha \in A\}$. Let $G_k^* = \bigcup_{i=1}^{\omega}\mathcal{G}_k$ for each $k$ and $\mathcal{G}^* = \bigcup_{k=1}^{\omega}G_k^*$. Define for each $i \geq 1$ and $j \geq 1$,

$$\mathcal{P}(i,j) = \{x \in X : \text{ord}(x,\mathcal{G}^*) < i \text{ or ord}(x,\mathcal{G}^*) = i \text{ and } 0 < \text{ord}(x,G_k) \leq j \text{ for some } k\}.$$

We show that for each $(i,j)$ there exists a sequence of collections $\{\mathcal{J}_k\}_{k=1}^{\omega}$ such that $\mathcal{J}_k$ is a discrete closed collection in $X - \mathcal{P}(i,j)$. Since $X = \bigcup_{i=1}^{\omega}\bigcup_{j=1}^{\omega}\mathcal{P}(i,j)$ and $\mathcal{P}(i,j+1) = \mathcal{P}(i,j) \cup [\bigcup_{k=1}^{\omega}[\bigcup_{j=1}^{\omega}\mathcal{J}_k]]$ the proof will be complete.

Let $i$ and $j$ be fixed.

Define, $H_i = \{x \in X : \text{ord}(x,\mathcal{G}^*) \leq i\}$.

$$B_k = \{B \subseteq A_k : |B| = j + 1\}.$$

$$S_k = \{x \in X : 0 < \text{ord}(x,\mathcal{G}_k) \leq j + 1\}.$$

Now for each $k$ and each $B \in B_k$ let $F(B,k) = [\bigcap_{\alpha \in B}G(\alpha,k)] \cap [G_k^* \cap H_i \cap S_k]$ and $\mathcal{J}_k = \{F(B,k) : B \in B_k\}$.

We assert that $\mathcal{J}_k$ is a discrete closed collection in $X - \mathcal{P}(i,j)$. Let $k$ be fixed and $x \in X - \mathcal{P}(i,j)$. Then $\text{ord}(x,\mathcal{G}^*) \geq i$.

(1) If $\text{ord}(x,\mathcal{G}^*) > i$, then $X - H_i$ is a neighborhood of $x$ which intersects no member of $\mathcal{J}_k$.

(2) Suppose $\text{ord}(x,\mathcal{G}^*) = i$.

**Case I.** If $x \notin G_k^*$, then $x$ belongs to exactly $i$ other members $\{G^*_\alpha : \alpha = 1,2,\cdots,i\}$ of $\mathcal{G}^*$. Hence $\bigcap_{\alpha=1}^{i}G^*_\alpha$ is a
neighborhood of \( x \) which misses \( G_k^* \cap H_i \) and hence intersects no member of \( J_k \).

Case II. Suppose \( x \in G_k^* \). If \( \text{ord}(x, \mathcal{G}_k) > j + 1 \) then \( x \) belongs to at least \( j + 2 \) members of \( \mathcal{G}_k \), say \( G(\alpha_\ell, k) \) for \( \ell = 1, \ldots, j+2 \). But \( \bigcap_{\ell=1}^{j+2} G(\alpha_\ell, k) \cap S_k = \emptyset \), so \( \bigcap_{\ell=1}^{j+2} G(\alpha_\ell, k) \) intersects no member of \( J_k \).

Finally if \( \text{ord}(x, \mathcal{G}_k) = j + 1 \) then \( x \) belongs to exactly \( j + 1 \) members of \( \mathcal{G}_k \), \( G(\alpha_\ell, k) \) for \( \ell = 1, 2, \ldots, j + 1 \). Then \( \bigcap_{\ell=1}^{j+1} G(\alpha_\ell, k) \) intersects only \( F(B, k) \) where \( B = \{\alpha_1, \alpha_2, \ldots, \alpha_{j+1}\} \).

It is easy to see that \( P(i, j+1) = P(i, j) \cup \bigcup_{k=1}^{\infty} \bigcup J_k \) so that the proof is complete. Hence \( X \) has property \( B(D, (\omega_0)^2) \).

Remark. It is important to note that in the construction above, the families \( J_k \) cover all points which have finite positive order with respect to some \( \mathcal{G}_k \).

Lemma. If \( \mathcal{U} \) be an open cover of a space \( X \) and \( C \) a closed subset of \( X \). Suppose that \( J = \{F_\alpha : \alpha \in A\} \) is a partial refinement of \( \mathcal{U} \) such that

1. each member of \( J \) is closed in \( X - C \) and
2. \( J \) is locally finite on \( X - C \).

Then there exists a sequence of open collections \( \{\mathcal{G}_i\}_{i=1}^\infty \) which partially refined \( \mathcal{U} \), such that each \( x \in [\bigcup J] - C \) has finite positive order with respect to some \( \mathcal{G}_k \). (In fact, \( \text{ord}(x, \mathcal{G}_k) = 1 \) for some \( k \).)

Proof. Now if \( \Gamma_n = \{B : B \subseteq A, |B| = n\} \), define \( H(B) = \bigcap_{\beta \in B} F_\beta \) for each \( B \in \Gamma_n \). Note that \( H(B) \subseteq U(B) \) for some \( U(B) \in \mathcal{U} \). Let \( \mathcal{G}_n = \{G(B) : B \in \Gamma_n\} \), where
Theorem 2.3. If a space $X$ has property $B(LF, (\omega_0)^2)$, then $X$ is weakly $\theta$-refinable.

Proof. Suppose $X$ has property $B(LF, (\omega_0)^2)$ and $\mathcal{U}$ is an open cover of $X$. Then there exists a collection of families $\{ \mathcal{J}_s : s < (\omega_0)^2 \}$ such that

(i) each member of $\mathcal{J}_s$ is closed in $X - \bigcup \{ \bigcup_{s' < s} \mathcal{J}_{s'} \}$,

(ii) $\bigcup \{ \bigcup_{s' < s} \mathcal{J}_{s'} \}$ is closed in $X$ for each $s$,

(iii) $\mathcal{J}_s$ is locally finite in $X - \bigcup \{ \bigcup_{s' < s} \mathcal{J}_{s'} \}$.

By the previous lemma, there exists for each $s$, a sequence $\{ \mathcal{G}_s^i \}_{i=1}^{\infty}$ of open collections such that each point $x \in \bigcup_{s' < s} \mathcal{J}_{s'}$ has finite positive order with respect to $\mathcal{G}_k$, for some $k$. Without loss of generality we may assume that each $\mathcal{G}_k^s$ is a partial refinement of $\mathcal{U}$. It is easy to see that $\{ \bigcup_{i < \omega_0} \bigcup_{s < (\omega_0)^2} \mathcal{G}_i^s \}$ is a weak $\theta$-refinement of $\mathcal{U}$, and hence $X$ is weakly $\theta$-refinable.

Remark. It should be noted that Theorem 2.3 above remains true for any countable ordinal $\beta$. The proof is similar.

Summary. Property $B(D, \omega_0) \Rightarrow$ weakly $\bar{\theta}$-refinable $\Rightarrow$ property $B(D, \omega_0)^2) \Rightarrow$ property $B(LF, (\omega_0)^2) \Rightarrow$ weakly $\theta$-refinable.
3. Property B (HCP, a ) and Irreducibility

In [17] the author obtained the following result.

Theorem 3.1. Every weak $\theta$-refinable space is irreducible.

Since property $B(D, \omega_0) \Rightarrow$ weakly $\theta$-refinable, every space with property $B(D, \omega_0)$ is irreducible. Here we can obtain the stronger result, that every space with property $B(HCP, a)$ is irreducible.

The following lemmas are straightforward, and hence their proofs are omitted.

Lemma 3.2. Let $H \subseteq X$ and let $U$ be a collection of open sets in $X$ which covers $H$. If $U\,|\,H$ has a minimal open (in $H$) refinement then there exists an open (in $X$) collection $V$ which partially refines $U$ and covers $H$, such that $V$ is a minimal open cover of $U\,V$.

Lemma 3.3. Let $X$ be a topological space and $H = \bigcup_{s < \alpha} H_s$ where $\bigcup_{s' < s} H_{s'}$ is a closed subset of $X$ for each $s < \alpha$. Let $U$ be a collection of open subsets of $X$ which covers $H$. If for each $s < \alpha$, $W_s$ is a collection of open subsets of $X$ which partially refines $U$ and covers $H_s - \bigcup_{s' < s} W_{s'}$ minimally, then there exists a collection $V$ of open subsets of $X$ which partially refines $U$, covers $H$, and is a minimal open cover of $U\,V$.

Theorem 3.4. Let $U = \{U_{a} : a \in A\}$ be a collection of open subsets of a space $X$ and $H = \{H_{a} : a \in A\}$ a hereditarily
closure preserving collection such that $H_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. Then $\mathcal{U}$ has an open partial refinement which covers $\cup H$ and is a minimal open cover of its union.

Proof. Suppose that $H = \{H_\alpha : \alpha \in A\}$ is a hereditarily closure preserving collection with $H_\alpha \subseteq U_\alpha$ for each $\alpha \in A$. We assume that $A$ is well ordered. For each $\alpha \in A$ choose

$$x_\alpha \in H_\alpha - \bigcup_{\beta < \alpha} H_\beta$$

when $H_\alpha - \bigcup_{\beta < \alpha} H_\beta \neq \emptyset$, and let $A' = \{ \alpha \in A : H_\alpha - \bigcup_{\beta < \alpha} H_\beta \neq \emptyset \}$. Since $X$ is $T_1$ and $H$ is hereditarily closure preserving $\{x_\alpha : \alpha \in A'\}$ is a discrete closed collection in $X$. Define

$$W_\alpha = U_\alpha - \{ x_\beta : \beta \in A' \text{ and } \beta \neq \alpha \}$$

for each $\alpha \in A$. Clearly $\mathcal{W} = \{ W_\alpha : \alpha \in A' \}$ is a minimal open cover of $\cup H$.

We now can obtain the following.

Theorem 3.5. Every space $X$ space with property $B(HCP,\alpha)$ is irreducible, for any ordinal $\alpha$.

Proof. Let $\mathcal{U}$ be an open cover of $X$. Then $\mathcal{U}$ has a refinement $\bigcup_{s < \alpha} \mathcal{V}_s$ satisfying properties in Definition 1.6 above. By induction we construct a sequence of $\{\mathcal{V}_s\}_{s < \alpha}$ of open collections such that for each $s < \alpha$,

(i) $\mathcal{V}_s$ is a partial refinement of $\mathcal{U}$,

(ii) $\bigcup_{s' < s} \mathcal{V}_{s'}$ covers $\bigcup_{s' < s} [\bigcup \mathcal{J}_{s'}]$

(iii) $\bigcup_{s' < s} \mathcal{V}_{s'}$ is a minimal open cover of its union.

(1) For $s = 1$, $\mathcal{J}_1$ is a hereditarily closure preserving collection of closed subsets of $X$. By Theorem 3.4 above there exists an open partial refinement $\mathcal{V}_1$ of $\mathcal{U}$ such that $\mathcal{V}_1$ is a minimal open cover of $\cup \mathcal{J}_1$. 


(2) Assume that \( V_{s'} \) has been constructed satisfying (i)-(iii) above for \( s' < s \). Define \( J^*_s = \{ F - \bigcup_{s' < s} V_{s'} \} : F \in J_s \} \) so that \( J^*_s \) is a hereditarily closure preserving collection in \( X \). By Theorem 3.4 again there exists an open partial refinement \( W_s \) of \( U \) such that \( W_s \) covers \( \bigcup J^*_s \) and is a minimal open cover of its union. Now define \( V_s = \{ W - \bigcup_{s' < s} V_{s'} \} : W \in W_s \} \). It is easy to check that \( V_s \) satisfies properties (i)-(iii) above and the induction is complete. As in Lemma 3.3 \( \bigcup V_s \) is a minimal open cover of \( X \) and refines \( U \). Hence \( X \) is irreducible.

**Corollary 3.6.** Every \( \mathfrak{N}_1 \)-compact space with property \( B(\text{HCP}, \alpha) \) is Lindelöf, where \( \alpha \) is any countable ordinal.

**Theorem 3.7.** Let \( f: X \to Y \) be a closed continuous map. If \( X \) has property \( B(\text{HCP}, \alpha) \), then \( Y \) has property \( B(\text{HCP}, \alpha) \) and hence is irreducible.

**Proof.** The proof follows from the fact that closure preserving collections are preserved under closed maps.

### 4. Applications and Shrinkability

**Definition 4.1.** An open cover \( \{ G_\alpha : \alpha \in A \} \) is shrinkable if there exists a closed cover \( \{ F_\alpha : \alpha \in A \} \) such that \( F_\alpha \subseteq G_\alpha \) for each \( \alpha \in A \).

In [19] the author obtained the following result.

**Theorem 4.2.** A space \( X \) is normal iff every weak \( \varnothing \)-cover of \( X \) is shrinkable.
A generalization of this result can now be proved using the notion of property above.

**Theorem 4.3.** Let \( \mathcal{G} = \{G_\alpha : \alpha \in A\} \) be an open cover of a space \( X \). If \( k \) is any countable ordinal, and \( \mathcal{G} \) has an open refinement \( \bigcup \mathcal{V}_s \) where \( \mathcal{V}_s = \{V(\alpha, s) : \alpha \in A\} \) satisfies,

1. \( V(\alpha, s) \subseteq G_\alpha \) for each \( \alpha \in A \),
2. \( \bigcup V(\alpha, s) \) is a cozero set in \( X \) for each \( s \),

then \( \mathcal{G} \) is shrinkable.

**Proof.** Define \( \mathcal{V}^*_s = \bigcup \mathcal{V}(\alpha, s) \) for each \( s < k \) so that \( \{\mathcal{V}^*_s : s < k\} \) is a countable cozero cover of \( X \). Then \( \{\mathcal{V}^*_s : s < k\} \) has a locally finite open refinement \( \{\mathcal{W}^*_s : s < k\} \) such that \( \mathcal{W}^*_s \subseteq \mathcal{V}^*_s \) for each \( s < k \). Define \( H(\alpha, s) = \mathcal{W}^*_s \cap V(\alpha, s) \) for each \( \alpha \in A \) and each \( s < k \), and \( H_\alpha = \bigcup H(\alpha, s) \). It should be clear that \( H_\alpha \subseteq G_\alpha \) for each \( \alpha \in A \) and \( \{H_\alpha : \alpha \in A\} \) covers \( X \). Hence \( \mathcal{G} \) is shrinkable.

**Theorem 4.4.** Let \( X \) be a normal space. For any countable ordinal \( k \), every open cover with property \( B(HCP,k) \) is shrinkable.

**Proof.** Let \( \mathcal{G} = \{G_\alpha : \alpha \in A\} \) be an open cover of \( X \) with property \( B(HCP,k) \) where \( k \) is any countable ordinal. Then \( \mathcal{G} \) has a refinement \( \bigcup \mathcal{J}_s \) where,

1. \( \mathcal{J}_s = \{F(\alpha, s) : \alpha \in A\} \) is HCP and closed in \( X - \bigcup_{s' < s} \mathcal{J}_{s'} \).
2. \( F(\alpha, s) \subseteq G_\alpha \) for each \( \alpha \in A \).

We show by transfinite induction that there exists for each \( s < k \), an open collection \( \mathcal{V}_s = \{V(\alpha, s) : \alpha \in A\} \) satisfying
(1) $V(\alpha, s) \subseteq \overline{V(\alpha, s)} \subseteq G_\alpha$ for each $\alpha \in A$,
(2) $\bigcup_{\alpha \in A} V(\alpha, s)$ is cozero in $X$ for each $s$.
(3) $\bigcup_{s' \in s} V_s$ covers $\bigcup_{s' \in s} J_s$ for each $s$.

Assume $V_s$ with the above properties has been constructed for all $s' < s$. Define $H(\alpha, s) = F(\alpha, s) - \bigcup_{s' \in s} V_s$ so that $H(\alpha, s) = \overline{H(\alpha, s)} \subseteq G_\alpha$ for each $\alpha \in A$. Since $H = \{H(\alpha, s) : \alpha \in A\}$ is closure preserving and $X$ is normal, there exists an open collection $\overline{V_s} = \{V(\alpha, s) : \alpha \in A\}$ such that $\overline{V_s}$ is a partial refinement of $\zeta$, and
(1) $H(\alpha, s) \subseteq V(\alpha, s) \subseteq \overline{V(\alpha, s)} \subseteq G_\alpha$ for each $\alpha \in A$,
(2) $\bigcup_{\alpha \in A} V(\alpha, s)$ is a cozero set in $X$.

Clearly $\bigcup_{s' \in s} V_s$ covers $\bigcup_{s' \in s} J_s$ and the construction is complete. By Theorem 4.3 above, $\zeta$ is shrinkable.

**Theorem 4.5.** Suppose that $X = \bigcup_{i=1}^{\infty} H_i$ where each $H_i = \overline{H_i}$ has property $B(D, \omega_0)$. Then $X$ has property $B(D, \omega_0)$.

**Proof.** Suppose each $H_i$ has property $B(D, \omega_0)$ and $\mathcal{U}$ is an open cover of $X$. Then $\mathcal{U}/H_i$ has a refinement $\mathcal{U}_{i=1}^{\infty} J_{j,k}^i$ such that $J_{j,k}^i$ is a discrete closed collection in $H_i = \bigcup_{k<j} J_{j,k}^i$.

Since $J_{j,k}^i$ is a discrete closed collection in $X$ for each $i$, the natural diagonalization of the families $\mathcal{U}_{i=1}^{\infty} J_{j,k}^i$ yields the desired collections satisfying property $F(D, \omega_0)$.

**Theorem 4.6.** Let $f : X \to Y$ be a perfect map.

(1) If $X$ has property $B(LF, \omega_0)$, then so does $Y$ and hence $Y$ is irreducible.

(2) If $X$ is weakly $\theta$-refinable, then $Y$ has property $B(LF, (\omega_0)^2)$ and hence is weak $\theta$-refinable.
Open Questions.

(1) Is weak $\theta$-refinability or weak $\theta$-refinability preserved under perfect or closed maps?

(2) Is metacompactness equivalent to weak $\theta$-refinable, almost expandable and orthocompactness?

(3) When are weakly $\theta$-refinable spaces irreducible?
For example, is countably metacompactness enough?

(4) When does property $B(D,(\omega_0)^2)$ imply weak $\theta$-refinability?

(5) Is there a simple example of a space which has property $B(D,\omega_0+1)$ but does not have property $B(D,\omega_0)$?

The author would like to thank the referee for his comments concerning this paper.

References


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