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In 1978 Paul R. Patten [P1] introduced the notions of a surjective approximate absolute retract (SAAR), a surjective approximate absolute neighborhood retract in the sense of Noguchi ($SAANR_N$), and a surjective approximate absolute neighborhood retract in the sense of Clapp ($SAANR_C$). He wanted to get generalizations of an absolute retract (AR) and an absolute neighborhood retract (ANR) which are locally connected and are therefore more natural than the original generalizations of Noguchi [N] and Clapp [C] if one wishes to study the properties of the image of a compact ANR under a certain class of maps (see [P2]).

The purpose of this note is to improve some results in [P1] and to prove some new results about Patten's notions.

Our results and presentation were greatly improved thanks to helpful comments by the referee.

We start by recalling the definitions of all generalizations of compact ANR's mentioned above. We assume that the reader is familiar with shape theory of compacta [B4].

We shall say that maps $f,g: Z \to Y$ of a space Z into a metric space (Y,d) are ε-*close* provided $d(f(z),g(z)) < \varepsilon$ for each $z \in Z$. If Z is a subset of Y and $f: Z \to Y$ is ε-close to the inclusion $i_{Z,Y}$ of Z into Y, then f is called an ε-*push*.

Definition. A map $r: Y \to X$ of a metric space Y into its subspace X is an *ε-retraction* provided $r|X$ is an ε-push. If, in addition, $r(X) = X$, then r will be called a *surjective ε-retraction.* When there is an ε-retraction of Y onto X for every $\varepsilon > 0$, we say that X is an *approximate retract* of Y. X is a *surjective approximate retract* of Y if for every $\varepsilon > 0$ there is a surjective ε-retraction of Y onto X.

Definition. By a (*surjective*) *approximate neighborhood retract of a metric space* Y *in the sense of Noguchi* will be meant a compact subspace X of Y which is a (surjective) approximate retract of some neighborhood of X in Y.

Definition. By a (*surjective*) *approximate neighborhood retract of a metric space* Y *in the sense of Clapp* will be meant a compact subspace X of Y such that for every $\varepsilon > 0$ there is a neighborhood U of X in Y and a (surjective) ε-retraction $r: U \to X$.

If a compact metric space X is a (surjective) approximate retract [a (surjective) approximate neighborhood retract in the sense of Noguchi or Clapp] of every metric space in which it is embedded, then X is said to be a (*surjective*) *approximate absolute retract* [a (*surjective*) *approximate absolute neighborhood retract in the sense of Noguchi* or *Clapp*]. These will be abbreviated as AAR (SAAR), [$AANR_N$ ($SAANR_N$), or $AANR_C$ ($SAANR_C$)], respectively.

It was observed in [P1] that every SAAR, $SAANR_N$, or $SAANR_C$ is locally connected (i.e., a Peano compactum) so that the class of SAAR, $SAANR_N$, or $SAANR_C$ is properly contained in the class of AAR, $AANR_N$, or $AANR_C$, respectively.

In the sequel we shall need the following proposition resembling (2.1) in [C] (see also [P1, Lemma 2]).

Proposition 1. *A compactum* X *in the Hilbert cube* Q *is an* SAAR (an SAANR_N or an SAANR_C) *iff* X *is a surjective approximate* (neighborhood) *retract of* Q (in the sense of Noguchi or Clapp, respectively).

Proof. Suppose that a compactum X in Q is a surjective approximate retract of Q and consider X also as a subset of a metric space M. Since every metric space can be embedded as a closed subset of an AR [Hu, p. 81] without loss of generality we can assume that M is an AR. Let $f: M \to Q$ be an extension of the identity map id_X and let $r: Q \to X$ be a surjective ε-retraction. Then $r \circ f: M \to X$ is also a surjective ε-retraction. Hence, X is an SAAR. The converse and the proofs of the remaining two statements are obvious.

Corollary 1. *Let* X *and* Y *be compacta such that* X *is a surjective approximate retract of* Y. *Then*

a) $Y \in$ AR *implies* $X \in$ SAAR; *and*

b) $Y \in$ ANR *implies* $X \in \mathrm{SAANR}_N$.

Proof. Consider X and Y as subsets of Q. The composition of a retraction of (a neighborhood N of Y in) Q and a surjective ε-retraction of Y into X will be a surjective ε-retraction of (the neighborhood N of X in) Q onto X. Hence, Proposition 1 applies.

An interesting problem is to decide whether a surjective approximate retract X of an SAAR (SAANR_C or SAANR_N) Y is an SAAR (SAANR_C or SAANR_N, respectively). The obvious attempt is to compose a surjective $(\varepsilon/2)$-retraction r of (a neighborhood N of Y in) Q onto Y and a surjective ($\varepsilon/2$)-retraction r_1 of Y onto X to get an ε-retraction of (the neighborhood N of X in) Q onto X. However, it is not clear that $r_1 \circ r(X) = X$. This will hold provided the following statement is true.

(*) Let Y and X, $X \subset Y$, be locally connected compacta. Then for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every δ-push $f: Y \to Y$ of Y onto itself, there is an ε-push $g_0: f(X) \to X$ of f(X) onto X which extends to a map $g: Y \to Y$.

The next two theorems show that the question as to whether an SAANR_C X is either an SAAR or an SAANR_N depends only on shape properties of X. The first of them improves Theorem 2 in [Pl] and resembles Theorem 7 in [Bo] while the second is similar to Theorem 8 in [Bo].

Theorem 1. *A compact metric space* X *is an* SAAR *iff* X *is both an* FAR *and an* SAANR_C.

Proof. Suppose X is an SAAR. Then X is also an AAR and therefore it is an SAANR_C and an FAR because Bogatyi [Bo] showed that FAR + $\text{AANR}_C \Leftrightarrow$ AAR.

Conversely, let X be both an FAR and an SAANR_C. We shall consider X as a subset of the Hilbert cube Q and prove that X is a surjective approximate retract of Q. This will clearly suffice by Corollary 1.

For a given $\varepsilon > 0$, since X is an SAANR_C, there is a neighborhood U of X in Q and a surjective ε-retraction $r: U \to X$. But, since X is also an FAR, by Borsuk's characterization of compacta with trivial shape [B3, Theorem (1.1)], there is a map $f\colon Q \to U$ such that $f|X = i_{X,U}$. The composition $R = r \circ f\colon Q \to X$ is a surjective ε-retraction of Q onto X. Hence, X is a surjective approximate retract of Q.

Theorem 2. *A compact metric space* X *is an* $SAANR_N$ *iff* X *is both an* FANR *and an* $SAANR_C$.

Proof. Since an $SAANR_N$ is clearly an $AANR_N$ and an $AANR_N$ is an FANR by [G], it follows that an $SAANR_N$ is both an FANR and an $SAANR_C$.

Conversely, suppose that X is an FANR and an $SAANR_C$. We shall assume that X is a subset of Q and prove that X is a surjective approximate neighborhood retract of Q in the sense of Noguchi. By Corollary 1, this would imply that X is an $SAANR_N$.

Since X is an FANR, there is a neighborhood V of X in Q such that for every neighborhood W of X in Q there is a homotopy $f_t^W\colon V \to Q$, $0 \leq t \leq 1$, satisfying $f_0^W = i_{V,Q}$, $f_1^W(V) \subset W$, and $f_1^W|X = i_{X,Q}$ [B4, p. 264]. On the other hand, since X is an $SAANR_C$, for each $\varepsilon > 0$ there is a neighborhood W_ε of X in Q and a surjective ε-retraction $r_\varepsilon\colon W_\varepsilon \to X$. But, the composition $r_\varepsilon \circ f_1^{W_\varepsilon}\colon V \to X$ is also a surjective ε-retraction. Hence, X is a surjective approximate neighborhood retract of Q in the sense of Noguchi.

Corollary 2. *Every one-dimensional* $SAANR_N$ *and every plane* $SAANR_N$ *is an* ANR.

Proof. Let X be a one-dimensional $SAANR_N$. Then X is locally connected and its first Betti number is finite because X is an FANR. Hence, X is an ANR [B2, p. 138].

Similarly, if X is a plane $SAANR_N$, then X is locally connected and its complement $R^2 - X$ has a finite number of components because X is an FANR. Hence, X is an ANR [B2, p. 138].

Corollary 3. *Every one-dimensional* SAAR *and every plane* SAAR *is an* AR.

Definition. Let $\mathcal{C}$ be a class of compacta. A compact metric space X is said to be *quasi-$\mathcal{C}$ space* if for every $\varepsilon > 0$ there exist a space $K \in \mathcal{C}$, a map f from X onto K, and a map g from K onto X such that $g \circ f$ is an ε-push.

Note that a compactum X is a quasi-ANR (a quasi-AR) [P1] iff X is quasi-$\mathcal{C}$ space when $\mathcal{C}$ is the class $\mathcal{ANR}$ ($\mathcal{AR}$) of all compact ANR's (AR's).

Proposition 2. *Let $\mathcal{C}$ and $\mathcal{D}$ be classes of compacta and let* X *be a quasi-$\mathcal{C}$ space.*

a) If each member of $\mathcal{C}$ is an SAAR, *then* X *is an* SAAR.

b) If each member of $\mathcal{C}$ is an $SAANR_C$, *then* X *is an* $SAANR_C$.

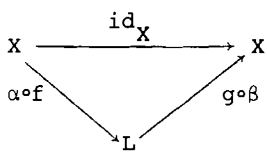
c) If each member of $\mathcal{C}$ is a quasi-$\mathcal{D}$ space, then X *is a quasi-$\mathcal{D}$ space.*

Proof of a). Without loss of generality we can assume that X and every member of $\mathcal{C}$ are compacta in Q. For an $\varepsilon > 0$ pick $K \in \mathcal{C}$ and maps $f: X \to K$ of X onto K and $g: K \to X$ of K onto X such that $g \circ f$ is an $(\varepsilon/2)$-push. Choose a

$\delta > 0$ such that g maps δ-close points into $(\varepsilon/2)$-close points. Let $F: Q \to Q$ be an extension of f and let $r: Q \to K$ be a surjective δ-retraction. It is easy to check that $g \circ r \circ F$ is a surjective ε-retraction of Q onto X. Hence, by Proposition 1, X is an SAAR.

The proof of b) is similar to the above proof of a).

Proof of c). First choose $K \in \mathcal{C}$ and maps $f: X \to K$ of X onto K and $g: K \to X$ of K onto X with $g \circ f$ $(\varepsilon/2)$-close to the id_X. Then pick $L \in \mathcal{D}$ and maps $\alpha: K \to L$ of K onto L and $\beta: L \to K$ of L onto K such that $\beta \circ \alpha$ is δ-close to id_K, where $\delta > 0$ has the property that g maps δ-close points of K into $(\varepsilon/2)$-close points of X. Clearly, the diagram



is ε-commutative. Hence, X is a quasi-$\mathcal{D}$ space.

In the investigation of properties of $AANR_C$'s the metric of continuity d_c [B1] played an important role [C]. For $SAANR_C$'s the metric $d_{\bar{c}}$ defined below has a similar role. It was introduced by Mazurkiewicz [M]. If $A, B \in 2^Y$ (i.e., A and B are nonempty compacta in a metric space (Y,d)) and there exist surjections from A to B and from B to A, then $d_{\bar{c}}(A,B)$ is the infimum of all $\varepsilon > 0$ for which there are ε-pushes of A onto B and B onto A.

Corollary 4. *Let $\mathcal{C}$ be a class of compacta, let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta in a metric space* Y, *and let* $\lim d_{\bar{c}}(A_n, A_0) = 0$. *If each* A_n $(n = 1,2,\cdots)$ *is a* *quasi-$\mathcal{C}$ space* (an SAAR, an $SAANR_C$), *then* A_0 *is also a quasi-$\mathcal{C}$ space* (an SAAR, an $SAANR_C$, respectively).

Proof. Since A_0 is a quasi-$\{A_1, A_2, \cdots\}$ space, we can apply Proposition 2.

Proposition 3. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta in an* ANR Y *and let* $\lim d_{\bar{C}}(A_n, A_0) = 0$. *If each* A_n $(n = 1,2,\cdots)$ *is a surjective approximate neighborhood retract of* Y *in the sense of Clapp, then* A_0 *is also a surjective approximate neighborhood retract of* Y *in the sense of Clapp*.

Proof. For an $\varepsilon > 0$, pick an index $n > 0$ such that $d_{\bar{C}}(A_n, A_0) < \varepsilon/3$. Let u and v be $(\varepsilon/3)$-pushes of A_0 onto A_n and A_n onto A_0, respectively. Since A_n is a surjective approximate neighborhood retract of Y in the sense of Clapp there is an open neighborhood V of A_n in Y for which there is a surjective $(\varepsilon/3)$-retraction $r: V \to A_n$. Let $u^*: V^* \to V$ be an extension of the map u to a neighborhood V^* of A_0 in Y such that u^* is also an $(\varepsilon/3)$-push. Then $v \circ r \circ u^*: V^* \to A_0$ is a surjective ε-retraction of V^* onto A_0.

Corollary 5. *Let* X *be a compact subset of* Q. *If there is a sequence* $\{P_n\}$ *of polyhedra in* Q *such that* $X = \lim P_n$ *in the metric* $d_{\bar{C}}$, *then* X *is both an* $SAANR_C$ *and a quasi*-ANR.

It is interesting that the converse of Corollary 5 is also true. In order to prove that (in Theorem 3 below) we shall need the following lemma which describes a useful property of an $SAANR_C$. Since a compactum is an $SAANR_C$ iff it has finitely many components each of which is an $SAANR_C$ it suffices to consider only a connected $SAANR_C$.

A map $f: X \to Y$ of a space X into a metric space (Y,d) is ε-*dense* if for every $y \in Y$ there is an $x \in X$ with $d(f(x),y) < \varepsilon$. We shall say that a Peano continuum Y is an S-*space* provided for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-dense map $f: X \to Y$ of a Peano continuum X into Y is ε-close to a map $f': X \to Y$ of X onto Y.

Lemma 1. *Every connected* $SAANR_C$ *is an* S-*space.*

Proof. Let B be a connected nondegenerate $SAANR_C$ and let an $\varepsilon > 0$ be given. Assume that B is a subset of Q and select a compact neighborhood U of B in Q for which there is a surjective $(\varepsilon/2)$-retraction $r: U \to B$. Pick the required $\delta > 0$ so that r maps 6δ-close points of U into $(\varepsilon/2)$-close points of B and so that the open 6δ-neighborhood $N_{6\delta}(B)$ of B lies in U.

Consider a δ-dense map $f: A \to B$ of a Peano continuum A into B. We can assume that the image $f(A)$ is nondegenerate (by taking a smaller δ if necessary). By Lemma 5 in [LM], there is a 6δ-push $h: f(A) \to Q$ such that $h(f(A)) \supset N_{2\delta}(f(A))$. But, since f is a δ-dense map into B, $N_\delta(B) \subset N_{2\delta}(f(A))$ so that $h(f(A)) \supset N_\delta(B)$. This implies that $f' = r \circ h \circ f: A \to B$ is a map of A onto B which is ε-close to f.

Lemma 2. *Let* $\mathcal{C}$ *be a class of* S-*spaces. Then every quasi-*$\mathcal{C}$ *continuum* X *is also an* S-*space.*

Proof. For a given $\varepsilon > 0$, pick an S-space $K \in \mathcal{C}$, a map f of X onto K, and a map g of K onto X such that $g \circ f$ is an $(\varepsilon/2)$-push of X onto itself. Let $\delta_1 > 0$ be such that g maps δ_1-close points into $(\varepsilon/2)$-close points. Since K is an S-space, there is an $\eta > 0$ with the property that every η-dense map of a Peano continuum into K is δ_1-close to a map onto K. Finally, the required $\delta > 0$ is chosen so that f maps δ-close points into η-close points.

The equivalence of (i) and (ii) in the next theorem was first proved in [P1, Theorem 1].

Theorem 3. *For a compact connected subset* X *of* Q *the following are equivalent.*

(i) X *is a quasi*-ANR.

(ii) X *is an* $SAANR_{\mathcal{C}}$.

(iii) X *is an* $AANR_C$ *and an* S-*space.*

(iv) There is a sequence $\{P_n\}$ *of connected polyhedra in* Q *such that* $X = \lim P_n$ *in the metric* $d_{\bar{C}}$.

Proof. Since (i) ⇒ (ii) by Proposition 2(b), (ii) ⇒ (iii) by Lemma 1, and (iv) ⇒ (i) by Corollary 5, it remains to see that (iii) ⇒ (iv). For that implication it suffices to show that given an $\varepsilon > 0$, there is a polyhedron P in Q such that $d_{\bar{C}}(X,P) < \varepsilon$. Let $\varepsilon > 0$ be given. Pick a $\delta, 0 < \delta < \varepsilon/4$, with respect to $\varepsilon/2$ using the fact that X is an S-space. Then we take an $\eta, 0 < \eta < \delta/2$, such that there is a δ-push $r\colon N_\eta(X) \to X$ [C, Lemma 4.2]. Next we select (see [HW, p. 73]) a connected polyhedron P^* and a map $f\colon X \to P^*$ of X onto P^* with the preimage of each point in P* of diameter < η (i.e., f is an η-map of X onto P*). According to [K, Theorem 9, §41], there is an embedding h: P* $\to$ Q such that the composition h $\circ$ f is an η-push. Hence, P = h(P*) is a polyhedron in Q contained in $N_\eta(X)$ and $r|P$: P $\to$ X is a δ-dense map of P into X. By the choice of δ, there is a map g: P $\to$ X of P onto X which is (ε/2)-close to $r|P$. But, since r is a δ-push, g is an ε-push of P onto X. Hence, $d_{\bar{c}}(X,P) < \varepsilon$.

Corollary 6. *A continuum* X *is an* SAAR *iff* X *is an* AAR *and an* S-*space.*

Corollary 7. *Let* M *be a separable metric space and let* $SAANR_C(M)$ *denote the hyperspace of all connected* $SAANR_C$*'s in* M *with the topology of* $d_{\bar{c}}$. *Then* $SAANR_C(M)$ *is a separable metric space.*

Proof. Since M can be emebeded into Q, $SAANR_C(M)$ is a subspace of $SAANR_C(Q)$. But, $SAANR_C(Q)$ is separable because finite polyhedra in Q form a separable dense subset (by Theorem 3).

That the polyhedra are dense is proved as follows (see Borsuk, Theory of retracts). Let G denote a countable dense subset of Homeo (Q), the group of all homeomorphisms of Q onto itself. Let P_g denote all finite nondegenerate polyhedra in Q that are geometrically realized in Q (considered as a standard subset of ℓ_2). Let RP_g denote the countable subset of P_g consisting of all polyhedra in P_g all vertices of which have all coordinates rational. It is easy to see that $G(RP_g) = \{g(p) | p \in RP_g, g \in G\}$ is a countable dense subset in P(Q), the collection of finite nondegenerate polyhedra in Q, with the topology of $d_{\bar{C}}$.

Corollary 8. *Let* X *be a connected* $SAANR_C$. *Then both the hyperspace* 2^X *of all nonempty compacta in* X *and the hyperspace* C(X) *of all nonempty subcontinua of* X *metrized by the Hausdorff metric are* SAAR*'s.*

Proof. Modify Clapp's proof in section 6 of [C] of a similar result for $AANR_C$'s by requiring that all maps are onto.

Recall that a compactum X is $\mathcal{C}$*-like*, where $\mathcal{C}$ is a class of compacta, if for every $\varepsilon > 0$ there is an ε-map of X onto some member of $\mathcal{C}$. If each member of a class $\mathcal{D}$ of compacta is $\mathcal{C}$-like, then we say that the class $\mathcal{D}$ is $\mathcal{C}$*-like.*

Proposition 4. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be classes of compacta and assume that* $\mathcal{D}$ *is* $\mathcal{C}$*-like. If an* $SAANR_C$ X *is a quasi-*$\mathcal{D}$ *space then* X *is also a quasi-*$\mathcal{C}$ *space.*

Proof. For a given $\varepsilon > 0$, pick a $K \in \mathcal{D}$, a map $f: X \to K$ of X onto K, and a map $g: K \to X$ of K onto X such that $g \circ f$ is an $(\varepsilon/2)$-push. Since X is a quasi-ANR by Theorem 3, we can use Lemma 1 in [P1] and get a $\delta > 0$ such that if u is a δ-map of K onto an $L \in \mathcal{C}$, then there is a map v of L onto X such that g is $(\varepsilon/2)$-close to $v \circ u$. It is easy to check that $v \circ (u \circ f)$ is an ε-push of X onto itself. Hence, X is a quasi-$\mathcal{C}$ space.

Proposition 5. *Let* $\mathcal{C}$ *be a class of continua. If a connected* $SAANR_C$ X *in* Q *is* $\mathcal{C}$*-like, then* $X = \lim K_i$ *in the metric* $d_{\bar{C}}$ *where each* K_i *is homeomorphic to some member of* $\mathcal{C}$.

Proof. The proof is almost identical to the proof of (iii) ⇒ (iv) in Theorem 3.

Corollary 9. *Let* q - $\mathcal{AR}$ *denote the class of all quasi*-AR'*s. If an* $SAANR_C$ *is* (q - $\mathcal{AR}$)-*like, then it is a quasi*-AR.

Proof. Combine Corollary 4 and Proposition 5. This corollary improves Theorem 3 in [P1].

Corollary 10. *If* f: X → Y *is a refinable map* [FR] *of an* $SAANR_C$ X *onto a quasi*-AR Y, *then* X *is also a quasi*-AR.

Proof. We can apply Corollary 9 because X is Y-like [FR].

The last two corollaries gave partial answers to Patten's question: is every SAAR a quasi-AR? The following proposition also gives a sufficient condition on an SAAR to be a quasi-AR.

Proposition 6. *Let* $X \subset Q$ *be an* SAAR *and assume that there is an equicontinuous family* $\{r_k: Q \to X_k\}_{k=1}^{\infty}$ *of retractions with* $X_k \subset N_{1/k}(X)$ *and* $X \subset X_k$ *for each* $k > 0$. *Then* X *is a quasi*-AR.

Proof. Let an $\varepsilon > 0$ be given. Let U be a compact neighborhood of X in Q for which there is an $(\varepsilon/2)$-push $r: U \to X$ satisfying $r(X) = X$. Then select a $\delta > 0$ such that r maps δ-close points of U into $(\varepsilon/2)$-close points of X. Next we pick an $\eta > 0$ and an index k_0 such that $N_{3\eta}(X) \subset U$ and r_k maps 3η-close points of Q into δ-close points for all $k \geqq k_0$. Since we can assume that X is a nondegenerate Peano continuum, according to Lemma 5 in [LM], there is a 3η-push $h: X \to Q$ such that $h(X) \supset N_\eta(X)$. Let $k \geqq k_0$ be such that $X_k \subset N_\eta(X)$. It is easy to check that $u = r_k \circ h$ is a map of X onto X_k and $v = r|X_k$ is a map of X_k onto X and that $v \circ u$ is an ε-push of X onto itself.

Proposition 7. *Let* $P = \Pi_{k>0} X_k$ *be the Cartesian product of continua* X_k. *Then*

a) P *is an* SAAR *is each* X_k *is an* SAAR;

b) P *is an* SAANR_C *if each* X_k *is an* SAANR_C; *and*

c) P *is an* SAANR_N *if each* X_k *is an* SAANR_N *and almost all* X_k*'s are* SAAR*'s*.

Proof of b). It is clear that the product of finitely many SAANR_C's is an SAANR_C. The statement for countably infinitely many factors follows from Proposition 2(b) because P is a quasi-$\mathcal{C}$ space, where $\mathcal{C} = \{X^1, X^2, \cdots\}$ is the set of all finite products $X^n = \Pi_{k=1}^n X_k$. Indeed, for $n > 0$, let $f_n: P \to X^{n+1}$ denote the projection and let $g_n: X^{n+1} \to P$ be a map defined by $g_n(x_1, \cdots, x_n, x_{n+1}) = (x_1, x_2, \cdots, x_n, \phi(x_{n+1})_{n+1}, \phi(x_{n+1})_{n+2}, \cdots)$ where $\phi: X_{n+1} \to \Pi_{k>n} X_k$ is a map of a Peano continuum X_{n+1} onto a Peano continuum $\Pi_{k>n} X_k$. The composition $g_n \circ f_n$ will be close to the id_P provided n is large enough.

Proof of a). Use b), Theorem 1, and [B4, p. 193].

Proof of c). Use b), Theorem 2, and [B4, p. 195].

Proposition 8. *Let* K(X) *denote the cone and* S(X) *the* (unreduced) *suspension of a continuum* X. *Then*

a) S(X) *is an* SAAR *if* X *is an* SAAR;

b) S(X) *is an* SAANR_C *if* X *is an* SAANR_C;

c) $S(X)$ *is an* $SAANR_N$ *if* X *is an* $SAANR_N$*; and*

d) $K(X)$ *is an* SAAR *if* X *is an* $SAANR_C$.

Proof. of b). Consider X as a subset of Q. By Theorem 3, there is a sequence $\{P_n\}$ of connected polyhedra in Q such that $X = \lim P_n$ in the metric $d_{\bar{C}}$. But, $\{S(P_n)\}$ is a sequence of connected polyhedra in the Hilbert cube $S(Q)$ [Ch] and $S(X) = \lim S(P_n)$ in the metric $d_{\bar{C}}$. Hence, by Theorem 3 again, $S(X)$ is an $SAANR_C$.

We shall close our remarks by identifying four interesting closed subsets of the hyperspace 2^Y of a metric space Y topologized by the metric $d_{\bar{C}}$ which are not closed in $(2^Y, d_C)$.

The first of them is provided by the class of all quasi-contractible compacta in Y. Recall [P1] that a compactum X is *quasi-contractible* provided for every $\varepsilon > 0$ there is an ε-push of X onto itself which is null-homotopic. Using Borsuk's Theorem (1.1) in [B3], it is easy to see that every quasi-contractible compactum has trivial shape. It follows from Theorem 2 in [P1] and our Theorem 1 that an $SAANR_C$ of trivial shape must be quasi-contractible.

Proposition 9. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta in a metric space* Y *and assume that* $\lim d_{\bar{C}}(A_n, A_0) = 0$. *If each* A_n $(n = 1,2,\cdots)$ *is quasi-contractible, then* A_0 *is also quasi-contractible.*

Proof. Let an $\varepsilon > 0$ be given. Select an index $n > 0$ so that $d_{\bar{C}}(A_n, A_0) < \varepsilon/4$. Let $f\colon A_0 \to A_n$ be an $(\varepsilon/4)$-push of A_0 onto A_n and let $g\colon A_n \to A_0$ be an $(\varepsilon/4)$-push of A_n

onto A_0. Observe that the composition $g \circ f$ is an $(\varepsilon/2)$-push of A_0 onto itself. Pick a $\delta > 0$ with the property that g maps δ-close points of A_n into $(\varepsilon/2)$-close points of A_0. Since A_n is quasi-contractible, there is an δ-push ϕ of A_n onto itself which is null-homotopic. Then $g \circ \phi \circ f$ is a null-homotopic ε-push of A_0 onto itself. Hence, A_0 is quasi-contractible.

The collection of all quasi-contractible compacta in Y is not closed in $(2^Y, d_c)$ because the closure of the $\sin(1/x)$-curve is a limit of arcs in the metric d_c and the closure of the $\sin(1/x)$-curve is a set of trivial shape which is not quasi-contractible.

Remark. The similar result can be proved for every property $\mathcal{P}$ of maps such that $\phi \in \mathcal{P}$ implies $g \circ \phi \circ f \in \mathcal{P}$. The examples of such properties besides "to be null-homotopic" are "to have category $\leqq k$," "to have cocategory $\leqq k$,"$\cdots$(see [Ga]).

The second closed subset of $(2^Y, d_{\bar{c}})$ is formed by all compacta A in Y having the surjective fixed point property (sfpp) (i.e., such that for every map f of A onto itself there is a point $a \in A$ with $f(a) = a$). Clearly, a compactum with the fpp also has the sfpp, but the converse does not hold (consider a disjoint union of an arc and a point).

Proposition 10. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of compacta in a metric space* Y *and assume that* $\lim d_{\bar{c}}(A_n, A_0) = 0$. *If each* A_n $(n = 1,2,\cdots)$ *has the* sfpp, *then* A_0 *also has the* sfpp.

Proof. Suppose u is a map of A_0 onto itself without fixed points. We shall find an index $n > 0$ and construct a map v of A_n onto itself without fixed points. This is an obvious contradiction to the assumption about A_n's.

Since a map u does not have fixed points, there is an $\varepsilon > 0$ such that $d(x,u(x)) \geqq 3\varepsilon$ for each $x \in A_0$. Pick an index $n > 0$ for which $d_{\bar{C}}(A_n,A_0) < \varepsilon$. Let f be an ε-push of A_0 onto A_n and let g be an ε-push of A_n onto A_0. Then the composition $v = f \circ u \circ g$ is a map of A_n onto itself without fixed points because $d(y,v(y)) \geqq \varepsilon$ for each $y \in A_n$.

The collection of all compacta in Y having the sfpp does not form a closed subset in $(2^Y,d_c)$ because the disjoint union of an arc and the Cantor set (which does not have the sfpp) is the limit of disjoint unions of an arc and finitely many points (that have the sfpp) in the metric of continuity.

Corollary 11. *Let* $\mathcal{C}$ *be a class of compacta with the* sfpp. *If an* $\text{SAANR}_{\mathcal{C}}$ X *is* $\mathcal{C}$*-like, then* X *has the* sfpp.

Proof. Combine Propositions 5 and 10.

It follows from the next proposition and the Lemma 3 below that the third closed subset of $(2^Y,d_{\bar{C}})$ provide all compacta in Y which quasi-embed into some space X. Recall that a compactum A is *quasi-embeddable* into a space X if for every $\varepsilon > 0$ there is an ε-map f of A into X (i.e., the diameter of each set $f^{-1}(x)$, $x \in f(A)$, is less than ε).

Lemma 3. *Let* $\mathcal{C}$ *be a class of compacta and suppose* $A_0 = \lim A_n$ *in the topology of* $d_{\bar{C}}$. *If* A_n *is* $\mathcal{C}$*-like for*

$n = 1,2,3,\cdots$, *then* A_0 *is* $\mathcal{C}$*-like.*

Proof. Obvious.

Note that the analogue of Lemma 3 for the metric d_c is false, as the example of a sequence of arcs converging to a point in the topology of d_c shows (using $\mathcal{C}$ as the class whose only member is an arc).

Proposition 11. *Let* $\mathcal{C}$ *be a class of compacta quasi-embeddable into a space* X. *If a compactum* A *is* $\mathcal{C}$*-like, then* A *is also quasi-embeddable into* X.

Proof. Let an $\varepsilon > 0$ be given. Pick a $K \in \mathcal{C}$ and an ε-map f of A onto K. Since K is compact, there is a $\delta > 0$ such that if Z is a subset of K with a diameter $< \delta$, then $f^{-1}(Z)$ has diameter $< \varepsilon$. Now, use the fact that K is quasi-embeddable into X and take a δ-map g of K into X. Clearly, $g \circ f$ is an ε-map of A into X.

The collection of all S-spaces in a metric space Y is the fourth example of a closed set in $(2^Y, d_{\bar{c}})$. This set is clearly not closed in $(2^Y, d_c)$ because a limit of Peano continua in the metric d_c need not be a Peano continuum.

Corollary 12. *Let* $\{A_n\}_{n=0}^{\infty}$ *be a sequence of continua in a metric space* Y *and assume that* $\lim d_{\bar{c}}(A_n, A_0) = 0$. *If each* A_n $(n = 1,2,\cdots)$ *is an* S*-space, then* A_0 *is also an* S*-space.*

Proof. Since A_0 is clearly a quasi-$\{A_1, A_2, \cdots\}$ space, we can apply Lemma 2.

Finally, let us observe that there is no evidence either in this note or in [P1] to suggest that the following conjecture is not true.

Conjecture. Every locally connected $AANR_C$ is an $SAANR_C$.

References

[Bo] S. A. Bogatyi, *Approximate and fundamental retracts* (in Russian), Mat. Sbornik 93 (135) (1974), 90-102.

[B1] K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fund. Math. 41 (1954), 168-202.

[B2] ________, *Theory of retracts*, Monografie Matematyczne 44, Warsaw, 1967.

[B3] ________, *Some remarks on shape properties of compacta*, Fund. Math. 85 (1974), 185-195.

[B4] ________, *Theory of shape*, Monografie Matematyczne 59, Warsaw, 1975.

[B5] ________, *On a class of compacta*, Houston J. Math. 1 (1975), 1-13.

[Ch] T. A. Chapman, *Lectures on Hilbert cube manifolds*, CBMS Regional Conferences Series in Math. #28, Providence, 1976.

[C] M. H. Clapp, *On a generalization of absolute neighborhood retracts*, Fund. Math. 70 (1971), 117-130.

[FR] J. Ford and J. W. Rogers, *Refinable maps*, Colloq. Math. 39 (1978), 263-269.

[Ga] T. Ganea, *Lusternik-Schnirelmann category and cocategory*, Proc. London Math. Soc. (3) 10 (1960), 623-639.

[G] A. Gmurczyk, *Approximative retracts and fundamental retracts*, Colloq. Math. 23 (1971), 61-63.

[Hu] S. T. Hu, *Theory of retracts*, Wayne State University Press, Detroit, 1965.

[HW] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, Princeton, 1948.

[K] K. Kuratowski, *Topology II*, Academic Press, New York, 1968.

[LM] I. Lončar and S. Mardešić, *A note on inverse sequences of* ANR'*s*, Glasnik Mat. (23) 3 (1963), 146-164.

[M] S. Mazurkiewicz, *Sur l'espace des continus pêaniens*, Fund. Math. 24 (1935), 118-134.

[N] H. Noguchi, *A generalization of absolute neighborhood retracts*, Kodai Math. Sem. Reports 1 (1953), 20-22.

[P1] P. R. Patten, *Locally connected generalized absolute neighborhood retracts*, Top. Proc. 3 (1978), 159-168.

[P2] ________, *Refinable maps and generalized absolute neighborhood retracts* (preprint).

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