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## LONGITUDINAL SURGERY ON COMPOSITE KNOTS

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## LONGITUDINAL SURGERY ON COMPOSITE KNOTS

Bradd Evans Clark

### 1. Introduction

It was 1962 when Lickorish in [5] demonstrated that all compact connected orientable 3-manifolds without boundary could be represented as a result of surgery on a link in  $S^3$ . Since then a great effort has been made to decide the Poincaré conjecture by considering surgery on knots and links. But surprisingly little else is known about the relationship between specific types of knots and the characteristics of the manifolds that can be obtained by surgery on those knots. This paper will demonstrate a relationship between knots and the Heegaard genus of manifolds.

Let  $X$  be a point set. We shall use  $\text{int}(X)$  for the interior of  $X$ ,  $\text{cl}(X)$  for the closure of  $X$ , and  $\partial X$  for the boundary of  $X$ . If  $K$  is a cube-with-knotted hole, the longitudinal curve of  $K$  will be the simple closed curve on  $\partial K$ , unique up to isotopy, which bounds an orientable surface in  $K$ . The meridional curve of  $K$  will be the simple closed curve on  $\partial K$ , unique up to isotopy, which is transverse to the longitudinal curve, and which bounds a disk in  $\text{cl}(S^3 - K)$ . The genus of a manifold is defined to be the minimal genus of a Heegaard splitting of the manifold. Let  $X$  be a polyhedron contained in the P.L.  $n$ -manifold  $M$ .  $N(X) \subset M$  is called a regular neighborhood of  $X$  in  $M$  if  $X \subset \text{int } N(X)$  and  $N(X)$  is an  $n$ -manifold which can be simplicially collapsed to  $X$ .

This paper deals with piecewise linear topology. As such, all manifolds are considered to be simplicial and maps to be piecewise linear. This will be assumed as additional hypotheses throughout the paper.

## 2. Algebraic Lower Bounds

Let  $k$  be an  $(m, 2)$  torus knot and let  $K$  be its associated cube-with-knotted hole. We shall use the notation  $nk$  to stand for the composition of  $n$  copies of  $k$ , and  $nK$  to stand for the cube-with-knotted hole associated with  $nk$ . Since  $\pi_1(K)$  has the form  $\{x_1, x_2 | r_2\}$ , we can use Van Kampen's Theorem to compute  $\pi_1(nK)$  by attaching each component of the composite in such a way that one of the generators of that component is identified with  $x_1$ . The result is that  $\pi_1(nK) \approx \{x_1, x_2, \dots, x_{n+1} | r_2, r_3, \dots, r_{n+1}\}$  where

$$r_i = (x_1 x_i)^{\frac{m-1}{2}} x_1 (x_i^{-1} x_1^{-1})^{\frac{m-1}{2}} x_i^{-1} \text{ for } 2 \leq i \leq n+1.$$

Goodrick in [4] was able to show that this is an  $n+1$  generator group.

Let  $M^3$  be the 3-manifold obtained from  $nK$  by sewing a solid torus  $T$  to  $\partial nK$  in such a way that the boundary of a meridional disk of  $T$  is sewn to the longitudinal curve of  $nK$ . We have that  $\pi_1(M^3) \approx \{x_1, x_2, \dots, x_{n+1} | r_2, r_3, \dots, r_{n+1}, L\}$  where  $L = L_2 L_3 \dots L_{n+1}$  and

$$L_i = x_i (x_1 x_i)^{\frac{m-3}{2}} x_1 (x_1 x_i)^{\frac{m-1}{2}} x_1^{2(1-m)} \text{ for } 2 \leq i \leq n+1.$$

*Proposition 2.1.*  $\pi_1(M^3)$  is an  $n+1$  generator group.

*Proof.* Longitudinal surgery on a knot yields a

3-manifold whose infinite cyclic covering has the same  $\mathbb{Z}[t, t^{-1}]$  module-structure as the infinite cyclic cover of the knot complement. This follows directly from the method of construction of such covers as demonstrated in [7]. Therefore all of the Alexander invariants of the longitudinal surgery manifold and the knot complement are identical. Since Goodrick in [4] used the elementary ideals of the Alexander matrix to prove that  $\pi_1(nK)$  was an  $n + 1$  generator group, it follows that  $\pi_1(M^3)$  is an  $n + 1$  generator group.

*Corollary 2.2.* If  $g$  is the Heegaard genus of  $M^3$  then  $g \geq n + 1$ .

*Proof.* Any Heegaard splitting of  $M^3$  can be associated with a natural presentation of  $\pi_1(M^3)$  which has as many generators as the genus of the splitting. But Proposition 2.1 shows that any presentation must have at least  $n + 1$  generators.

### 3. Geometric Upper Bounds

An upper bound to the Heegaard genus of any 3-manifold obtained by surgery on a link can be found by using the "folklore" concept of a tunnel number.

*Definition.*  $t \subset K$  is a *tunnel* if  $t$  is a 3-cell and  $t \cap \partial K$  is a pair of disjoint 2-cells.

It is well established that if  $K$  is a cube-with-knotted hole, then it is possible to remove a series of tunnels from  $K$  and thereby convert  $K$  into a handlebody.

*Definition.* If  $L$  is an unsplitable link, the *tunnel number* of  $L$ ,  $T(L)$ , is the minimum number of disjoint tunnels needed to convert  $\text{cl}(S^3 - N(L))$  into a handlebody.

*Proposition 3.1.* Let  $nk$  be the composition of  $n$  copies of an  $(m,2)$  torus knot and let  $nK = \text{cl}(S^3 - N(nk))$ . The tunnel number of  $nk$  is  $n$ .

*Proof.* In [1] it was shown that any torus knot has tunnel number 1. We can find a set of  $n$  2-spheres  $\{S_1, S_2, \dots, S_n\}$  such that  $S_i \cap nk$  is a pair of points, and one component  $C_i$  of  $S^3 - S_i$  contains the  $i$ -th component of  $nk$ . We then can remove a single tunnel  $t_i$  from  $C_i \cap nk$  with  $t_i \cap S_i = \emptyset$  for  $1 \leq i \leq n$  and obtain a handlebody of genus  $n + 1$ .

The traditional approach to building a 3-manifold of high Heegaard genus is to take the Cartesian product of a closed 2-manifold of high genus with a circle. A quick glance at the homology of such a manifold shows that it cannot be obtained by surgery on a knot. However, the next theorem will demonstrate that manifolds of any Heegaard genus can be generated by surgery on knots.

*Theorem 3.2.* Let  $n \geq 0$  be an integer. There exist infinitely many knots  $k$  such that surgery on  $k$  will yield a manifold of Heegaard genus  $n$ .

*Proof.* Trivial surgery on any knot will, of course, yield  $S^3$ , a manifold of genus 0. Louise Moser in [6] showed that if  $k$  is a torus knot of type  $(r,s)$  and  $(p,q)$  surgery is performed with  $|rsp + q| = 1$ , then the manifold obtained

is a lens space. It was shown in [1] that any other type of surgery on a torus knot yields a 3-manifold of genus 2.

Now suppose  $n \geq 3$  and  $(n - 1)k$  is the composition of  $(n - 1)$  copies of an  $(m, 2)$  torus knot. We know by Corollary 2.2 that longitudinal surgery on  $(n - 1)k$  will yield a manifold of Heegaard genus at least  $n$ . By Proposition 3.1 we know that we can find a set  $\{t_1, t_2, \dots, t_{n-1}\}$  of pairwise disjoint tunnels that by removal will transform  $cl(S^3 - N((n-1)k)) = (n - 1)K$  into the handlebody  $H$ .

If we perform surgery on  $(n - 1)k$ , we have that the image,  $h(N((n - 1)k))$ , in the resulting surgery manifold  $M^3$ , is still a solid torus. Thus  $h(N((n - 1)k) \cup t_1 \cup \dots \cup t_{n-1})$  is a handlebody  $H'$  of genus  $n$ . But  $cl(M^3 - H') = H$  since  $H$  is unaffected by the surgery on  $(n - 1)k$ . Thus any manifold obtained by surgery on  $(n - 1)k$  has a Heegaard splitting of genus  $n$ .

It is readily seen that this concept of a tunnel number can yield an upper bound on the Heegaard genus of any manifold obtained by surgery on a link. This idea was developed in [2], where it was shown that the tunnel number of knots and links acts differently from other invariants, such as the bridge number. Also, in the examples we have seen thus far, this maximal Heegaard genus is obtained when longitudinal surgery is performed on the knot.

*Question.* Does longitudinal surgery on  $k$  always yield a manifold of maximal Heegaard genus among those that can be formed by surgery on  $k$ ?

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