PRODUCTS OF SPACES OF COUNTABLE TIGHTNESS

by

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Introduction

As is well known, the product $X^2$ of a space $X$ of countable tightness need not have countable tightness. Also if $X$ is a CW-complex, $X^2$ is not always a CW-complex.

In this paper, in the first section, we consider the products of spaces of countable tightness and $k$-spaces. In the second section, we consider the products and the metrizability of CW-complexes.

1. Products of $k$-Spaces and Spaces of Countable Tightness

All spaces are assumed to be regular and $T_1$. We consider cardinals to be initial ordinals, and let $c$ denote the cardinality of the continuum. Let $N$ be the set of natural numbers.

We need the following well known example. This example will play an important role in the products.

Let $\alpha$ be an infinite cardinal number. Let $S_\alpha$ be the space obtained from the disjoint union of $\alpha$ convergent sequences by identifying all the limit points. $S_\omega$ is especially called the sequential fan.

We now recall some basic definitions.

Let $X$ be a space, and let $J = \{F_\gamma : \gamma \in \Gamma\}$ be a closed covering of $X$. Then $X$ has the weak topology with respect to $J$, if $F \subseteq X$ is closed whenever $F \cap F_\gamma$ is closed in $X$ for each $\gamma \in \Gamma$. 

A space $X$ is a *k-space* (resp. *sequential space*), if $X$ has the weak topology with respect to the collection of all compact subsets (resp. compact metric subsets) of $X$.

A space $X$ is a *$k_\omega$-space* [11], if it has the weak topology with respect to a countable covering of compact subsets of $X$.

A space $X$ has *countable tightness*, $t(X) \leq \omega$, if $x \in \overline{A}$ in $X$, then $x \in \overline{C}$ for some countable $C \subseteq A$. It is known that every sequential space has countable tightness.

**Proposition 1.1.** (1) If $X \times S_\subseteq C$ is a k-space, then each closed, separable subset of $X$ is locally countably compact.

(2) If $X \times S_\subseteq C$ has countable tightness, then each $k_\omega$-subspace of $X$ is locally compact.

**Proof.** (1) Suppose that there exists a closed, separable subset $S$ of $X$ which is not locally countably compact. Since $S$ is regular and $T_1$, as is well known, the weight of $S$ is equal or less than $c$. Hence some $x_0 \in S$ has a local base $\{U_\alpha : \alpha < m\}$ in $S$, $\omega < m \leq c$, such that each $\overline{U_\alpha}$ is not countably compact.

We now use the idea of E. Michael [10; Theorem 2.1]. For $\alpha < m$, since $\overline{U_\alpha}$ is not countably compact, there is a decreasing sequence $\{F_\alpha n : n \in N\}$ of non-empty closed subsets of $\overline{U_\alpha}$ with $\bigcap F_\alpha n = \emptyset$. Let $T_\alpha = \bigcup F_\alpha n \times n_\gamma$, where $n_\gamma$ denotes the $n$-th term of the $\alpha$-th sequence in $S_m$ and let $T = \bigcup_{\alpha < m} T_\alpha$. Then for each compact subset $K$ of $S \times S_m$, $T \cap K$ is closed in $S \times S_m$, because $K$ meets only finitely many $T_\alpha$'s and each $K \cap T_\alpha$ is a finite union of closed subsets of $S \times S_m$. But $T$ is not closed in $S \times S_m$. This
implies that \( S \times S_m \) is not a k-space. Since \( S \times S_m \) is a closed subset of \( X \times S_c \), \( X \times S_c \) is not a k-space. This is a contradiction.

(2) If a space has countable tightness, so does every subspace. Thus we may assume that \( X \) is a k\( _\omega \)-space. Since \( t(X \times S_c) \leq \omega \), \( X \times S_c \) has the weak topology with respect to the covering of all closed separable subsets of \( X \times S_c \). Since each subset \( S \) of \( X \times S_c \) is contained in \( X \times \pi(S)_c \), where \( \pi: X \times S_c \to S_c \) is the projection, \( X \times S_c \) has the weak topology with respect to a closed covering \( \{ X \times F; F \) is a closed separable subset of \( S_c \} \). Since we can assume that each \( F \) is contained in some \( S_\alpha \), \( \alpha < \omega_1 \), \( F \) is a k\( _\omega \)-space. By \([11; (7.5)]\), each \( X \times F \) is a k-space. Thus \( X \times S_c \) is a k-space. Hence, by (1) each closed, separable subset of \( X \) is locally countably compact.

We now show that \( X \) is locally compact. Let \( X \) have the weak topology with respect to a countable covering of compact subsets \( X_i \) with \( X_i \subseteq X_{i+1} \). For some \( x_o \in X \), suppose \( x_o \notin X - X_i \) for each \( i \). Since \( t(x) \leq \omega \), there are countable subsets \( C_i \subseteq X - X_i \) with \( x_o \in \overline{C_i} \). Let \( C = \bigcup_{i=1}^{\infty} C_i \). Then \( x_o \notin C \cap (X - X_i) \) for each \( i \). Since the closed separable subset \( C \) of \( X \) is locally countably compact, there exists a countably compact subset \( K \) of \( C \) such that \( x_o \in K \cap (X - X_i) \) for each \( i \). Since \( K \) is countably compact in \( X \), it is easy to see that \( K \) is contained in some \( X_{i_0} \). But

\[
x_o \in K \cap (X - X_{i_0}) = \emptyset.
\]

This is a contradiction. Thus each point of \( X \) is contained in some \( \text{int} \ X_i \). Hence \( X \) is locally compact.
A space $X$ is strongly Fréchet [14], i.e. countably bi-sequential due to E. Michael [12], if $x \in \overline{A_n}$ with $A_{n+1} \subseteq A_n$ then there exist $x_n \in A_n$ such that $x_n \to x$. If the $A_n$ are all the same set, then such a space $X$ is Fréchet.

Lemma 1.2. (cf. [15; 16(b) and p. 35]). Every Fréchet space which is not strongly Fréchet contains a copy of $S_\omega$.

Recall that a space $X$ is symmetric if there is a real valued, non-negative function $d$ defined on $X \times X$ satisfying the conditions:

1. $d(x,y) = 0$ whenever $x = y$;
2. $d(x,y) = d(y,x)$;

and (3) $A \subseteq X$ is closed in $X$ whenever $d(x,A) > 0$ for any $x \in X - A$. If we replace the condition (3) by the following: For $A \subseteq X$, $x \in \overline{A}$ if and only if $d(x,A) = 0$, then such a space is called semi-metric.

Corollary 1.3. Suppose $X \times S_c$ has countable tightness.

1. If $X$ is Fréchet, then $X$ is strongly Fréchet.
2. [CH]. If $X$ is symmetric, then $X$ is semi-metric.

When $X$ is paracompact, [CH] can be omitted.

Proof. (1) This follows from Proposition 1.1(2) and Lemma 1.2.

(2) Let $X$ be a symmetric space. Every Fréchet and symmetric space is first-countable ([1; p. 129]), hence is semi-metric. So, we prove that $X$ is Fréchet. To prove this, since $t(X) \leq \omega$, it is sufficient to show that every closed, separable subset $S$ of $X$ is first countable. Since
S is regular and $T_1$, each point of S has a local base of cardinality $\leq c$ in S. Then, under $\text{CH}$ each point of S is a $G_\delta$-set in S by [16; Theorem 10]. When $X$ is a paracompact space, without $[\text{CH}]$, the separable space S is Lindelöf. Thus, by [13; Theorem 2] S is hereditarily Lindelöf. Then each point of S is a $G_\delta$-set in S. Hence, then in any case each point of S is a $G_\delta$-set in S. Thus, by Proposition 1.1(2) and [8; Lemma 6.11], S is first countable.

A bi-$k$-space, according to E. Michael [12], is characterized as a bi-quotient image of a paracompact $M$-space. For the intrinsic definition of a bi-$k$-space, see [12; Definition 3.E.1].

Corollary 1.4. Suppose $f: X \to Y$ is a closed map with $t(Y) \leq \omega$. Let $X$ be a paracompact bi-$k$-space (resp. paracompact locally compact space). Then $Y \times S_C$ is a k-space (resp. $t(Y \times S_C) \leq \omega$) if and only if $Y$ is locally compact.

Proof. Let $Y$ be locally compact. Then $Y \times S_C$ is a k-space (resp. $t(Y \times S_C) \leq \omega$) by [3; 3.2] (resp. [9; Theorem 4]. So we prove the "only if" part. Suppose $Y \times S_C$ is a k-space. Then, by Proposition 1.1(1), $Y$ has property (P): Every closed separable subset is locally countably compact. Then, since $t(Y) \leq \omega$, it is easy to see that $Y$ satisfies Lemma 9.1(b) in [12]. Indeed, if $\{F_n: n \in \mathbb{N}\}$ is a decreasing sequence with $y \in \cap (\overline{F_n} - \{y\})$, then there exist $y_n \in F_n$ such that $\{y_n: n \in \mathbb{N}\}$ is not closed in $Y$. Then, by [12; Theorem 9.9], each $\partial f^{-1}(y)$ is compact. Thus, by [12; Proposition 3.E.4], $Y$ is a bi-$k$-space.
Next, we prove that $Y$ is locally compact. Suppose not. Then there is a point $y_0 \in Y$ such that $y_0 \notin \overline{Y - K}$ for every compact subset $K$ of $Y$. Let $J = \{X - K; K \text{ is compact in } Y\}$. Then $J$ is a filter base accumulating at the point $y_0$.

Since $Y$ is bi-$k$, by [12; Lemma 3.E.2] there is a decreasing closed sequence $\{A_n; n \in \mathbb{N}\}$ satisfying the following:

(a) $C = \cap A_n$ is compact;

(b) If $V$ is an open subset of $Y$ with $C \subseteq V$, then $C \subseteq A_n \subseteq V$ for some $n$; and

(c) $y_0 \in \overline{F \cap A_n}$ for all $n \in \mathbb{N}$ and all $F \in J$.

To prove some $A_n$ is compact, suppose not. Since $Y$ is paracompact, each $A_n$ is not countably compact. Then there are closed discrete subsets $D_n$ of $A_n$ with $|D_n| = \omega$.

Let $Y_0 = C \cup \bigcup_{n=1}^{\infty} D_n$ be a subspace of $Y$. Then $Y_0$ is closed in $Y$. Let $Z$ be a quotient space obtained from $Y_0$ by identifying the compact subset $C$. Then, by (a) and (b), $Z$ is not locally countably compact. Since $Y_0$ satisfies (P) and the countable space $Z$ is the perfect image of a closed separable subset of $Y_0$, so then $Z$ is locally countably compact. This is a contradiction. Hence some $A_{y_0}$ is compact.

But, by (c), $y_0 \in \overline{F \cap A_{y_0}} = \emptyset$. This is a contradiction.

Hence $Y$ is locally compact.

Finally we prove the parenthetical part. Let $t(Y \times S) \leq \omega$ and let $T$ be any closed separable subset of $Y$. Then $T$ is a closed image of a closed separable subset $S$ of $X$. Since $X$ is paracompact, $S$ is Lindelöf. Since $X$ is locally compact, it is easy to see that $S$ is a $k_\omega$-space.
Thus, since \( T \) is a quotient image of \( S \), \( T \) is also a \( k_\omega \)-space. Then, by Proposition 1.1(2), \( T \) is locally compact. Hence \( Y \) has Property (P). Thus, since \( t(Y) \leq \omega \), \( Y \) satisfies Lemma 9.1(b) in [12]. So, by [12; Theorem 9.9] each \( \partial f^{-1}(y) \) is compact. Thus \( Y \) is locally compact.

Let \( \alpha \) be an infinite cardinal. Recall that a space \( X \) is \( \alpha \)-compact if every subset of \( X \) of cardinality \( \alpha \) has an accumulation point in \( X \).

**Lemma 1.5.** Let \( f: X \to Y \) be a closed map with \( X \) collectionwise normal and \( Y \) sequential. If \( Y \) contains no closed copy of \( S_\alpha \), then each \( \partial f^{-1}(y) \) is \( \alpha \)-compact.

**Proof.** Suppose some \( \partial f^{-1}(y_0) \) is not \( \alpha \)-compact. Then there exists a closed discrete subset \( D \) of \( \partial f^{-1}(y_0) \) with \( |D| = \alpha \). Hence there is a discrete open collection \( \{V_d; d \in D\} \) of \( X \) with \( V_d \ni d \). For each \( d \in D \), since \( y_0 \in \overline{f(V_d)} - \{y_0\} \), \( y_0 \) is not isolated in a sequential space \( f(V_d) \). So then there is a sequence \( C_d = \{y_{dn}; n \in \mathbb{N}\} \) such that \( y_{dn} \in y_0 \) and \( C_d \subseteq \overline{f(V_d)} - \{y_0\} \). Since \( \{f(V_d); d \in D\} \) is hereditarily closure preserving, so is the collection \( C = \{C_d \cup \{y_0\}; d \in D\} \). Let \( Y_0 \) be the union of \( C \). Then \( Y_0 \) is closed in \( Y \). Let \( Z \) be the disjoint union of \( C \), and let \( g: Z \to Y_0 \) be the obvious map. Then \( Z \) is metric and \( g \) is closed with \( \partial g^{-1}(y_0) \) not \( \alpha \)-compact. Hence, by [7; Lemma 2], \( Y_0 \) contains a closed copy of \( S_\alpha \). Thus \( Y \) contains a closed copy of \( S_\alpha \). This is a contradiction.

From Proposition 1.1(2) and Lemma 1.5, we have
Corollary 1.6. Let \( f: X \to Y \) be a closed map with \( X \) paracompact sequential. If \( t(Y \times S_c) \leq \omega \), then each \( \partial f^{-1}(y) \) is compact.

By Lemma 1.5, we can generalize all results in this section as follows.

Generalization. Let \( S \) be a sequential space which is a closed image of a collectionwise normal space under \( f \) such that some \( \partial f^{-1}(s) \) is not c-compact. Then, for all results in this section we can replace "\( S_c \)" by "\( S \)."

By this generalization, for example we have the following:

Let \( Y \) be a Fréchet space. Let \( X \) be a collectionwise normal sequential space, and let \( F \) be a closed subset of \( X \). Suppose \( Z \) is a quotient space obtained from \( X \) identifying \( F \). Then \( Y \) is strongly Fréchet or \( \partial F \) is c-compact, if \( t(Y \times Z) \leq \omega \).

2. CW-Complexes

The concept of CW-complexes due to J. H. C. Whitehead [17] is well-known. We recall some basic properties of CW-complexes. Let \( X \) be a CW-complex; that is, \( X \) is a complex which is closure finite (i.e. each cell of \( X \) is contained in a finite subcomplex), and which has the weak topology with respect to the closed covering \( \{ L_\gamma ; \gamma \in \Gamma \} \) of all finite subcomplexes \( L_\gamma \) of \( X \). Then for any subset \( \Gamma' \) of \( \Gamma \), \( L' = \bigcup_{\gamma \in \Gamma'} L_\gamma \) is closed in \( X \) and \( L' \) has the weak topology with respect to a closed covering \( \{ L'_\gamma ; \gamma \in \Gamma' \} \).
As a topological complex, C. H. Dowker [4] introduced the concept of the Whitehead complex. A space $X$ is a Whitehead complex, if it is an affine complex (see [4; §1]) having the weak topology with respect to $\{e_\lambda; \lambda\}$. Here $\{e_\lambda; \lambda\}$ is the cells of $X$. Recall that the closure $\overline{e_\lambda}$ of $e_\lambda$ coincides with the topological closure in $X$ of $e_\lambda$ [4; p. 560], and this also holds in CW-complexes. Every Whitehead complex with the cells $\{e_\lambda; \lambda\}$ is a CW-complex with each $e_\lambda$ a subcomplex [4; p. 558].

We need the canonical example $S_2$ due to S. P. Franklin [5; Example 5.1]. That is, $S_2 = (N \times N) \cup N \cup \{0\}$ with each point of $N \times N$ is an isolated point. A basis of neighborhoods of $n \in N$ consists of all sets of the form $\{n\} \cup \{(m,n); m \geq m_0\}$. And $U$ is a neighborhood of $0$ if and only if $0 \in U$ and $U$ is a neighborhood of all but finitely many $n \in N$.

Lemma 2.1. Suppose that $X$ has the weak topology with respect to a point-countable closed covering $\{C_\alpha; \alpha\}$ of $X$.

(1) Let each $C_\alpha$ be Fréchet. Then $X$ is Fréchet if and only if $X$ contains no copy of $S_2$.

(2) Let each $C_\alpha$ be metric. Then $X$ is metric if and only if $X$ is a paracompact, strongly Fréchet space.

Proof. (1) Since $S_2$ is not Fréchet, the "only if" part follows from that every subset of a Fréchet space is Fréchet.

We prove the "if" part. Suppose $X$ is not Fréchet. Since $X$ is sequential, by [5; Proposition 7.3] $X$ contains
a subspace \( M = (N \times N) \cup N \cup \{0\} \) which, with the sequential closure topology, is a copy of \( S_2 \). The countable space \( M \) intersects at most countably many \( C_\alpha \)'s, say \( C_{\alpha_1}, C_{\alpha_2}, \ldots \). Let \( X_n = \bigcup_{i=1}^{n} C_{\alpha_i} \) and let \( C \) be a compact subset of \( M \). Then \( C \) has the weak topology with respect to a countable closed covering \( \{X_n \cap C; \ n \in N\} \) of \( C \). Hence \( C \) is contained in some \( X_n \cap C \). Thus each convergent sequence in \( M \) is contained in some \( X_n \). We also remark that each \( X_n \) is Fréchet, hence contains no copy of \( M \).

We now use the method of proof of S. P. Franklin and B. V. Smith Thomas [6; Proposition 1]. Since \( N \cup \{0\} \) is a convergent sequence in \( M \), there is \( X_{n_0} \) with \( N \cup \{0\} \subseteq X_{n_0} \). Let \( C_n = \{n\} \times N \cup \{n\} \) for each \( n \). Since \( C_1 \) is a convergent sequence, there is \( X_{n_1} \) \((n_1 > n_0)\) with \( C_1 \subseteq X_{n_1} \).

Since \( X_{n_1} \) is closed and Fréchet, we can choose \( C_{n_2} \) \((n_2 > 1)\) and \( X_{n_3} \) \((n_3 > n_2)\) such that \( C_{n_2} \cap X_{n_1} \) is at most finite and \( C_{n_2} \subseteq X_{n_3} \). So, we can assume that \( C_{n_2} \subseteq X_{n_3} - X_{n_1} \). In this way, we can choose \( C_{n_k} \) and \( X_{n_{k+1}} \) \((n_k+1 > n_k > n_{k-1})\) with \( C_{n_k} \subseteq X_{n_{k+1}} - X_{n_{k-1}} \). Let \( M' = \bigcup_{k=1}^{\infty} C_{n_k} \cup \{n_k; k \in N\} \cup \{0\} \).

Then, for each \( \alpha \in \Lambda \), \( M' \cap C_\alpha \) is closed in \( X \). Thus \( M' \) is a closed subset of \( X \). Then \( M' \) is sequential, hence \( M' \) has the sequential closure topology. Thus \( M' \) is a copy of \( S_2 \). Hence \( X \) contains a copy of \( S_2 \). This is a contradiction.

(2) We prove only the "if" part. For \( x \in X \) let \( \{C_\alpha; \ C_\alpha \ni x\} \) be \( \{C_{\alpha_1}, C_{\alpha_2}, \ldots\} \). Put \( X_n = \bigcup_{i=1}^{n} C_{\alpha_i} \). Suppose
x ∈ X - X_n for each n. Since X is strongly Fréchet, there exist x_n ∈ X_n such that x_n → x.

Let C = \{x_n; n ∈ N\} U \{x\}. Then the compact subset C has the weak topology with respect to a countable covering \{C ∩ C_α; C ∩ C_α ≠ ∅\} of C. Then C is contained in some finite union of C_α. Thus some C_α must contain infinitely many x_n's, hence C_α ∋ x. Then C_α is contained in some X_n. But this is a contradiction, for X_n ∋ x_n for n ≥ n_0. Thus x ∉ X - X_n for some n, hence x ∈ int X_n. This implies that X is locally metrizable. Hence X is metrizable, for X is paracompact.

Lemma 2.2. Let X be a CW-complex with the cells \{e_γ\}. If X contains no closed copy of S_α, then for each x ∈ X the cardinality of \(Γ_x = \{γ; e_γ ∋ x\}\) is less than α.

Proof. For some x_o ∈ X, suppose \(|Γ_x| ≥ α\). Since e_γ ∈ x_o for γ ∈ Γ_x, there exist x_γn such that x_γn → x_o and x_γn ∈ e_γ. Let C_γ = \{x_γn; n ∈ N\} U \{x_o\} and let S = U{C_γ; γ ∈ Γ_x}. Suppose L is any finite subcomplex of X. Then S ∩ L is closed in X. Thus S is closed in X. Moreover S has the weak topology with respect to \{C_γ; γ ∈ Γ_x\}. Indeed, for F ⊆ S, let F ∩ C_γ be closed in S for each γ ∈ Γ_x. Then F ∩ L = \{F ∩ C_γ; e_γ ⊆ L and γ ∈ Γ_x\}. Thus F ∩ L is closed in S. Hence, F is closed in S. This implies that X contains a closed copy of S_α. This is a contradiction.

In [6], S. P. Franklin and B. V. Smith Thomas proved that a k_ω-space with metrizable "pieces" is metrizable if
and only if it contains no copy of $S_2$ and no sequential fan $S_\omega$.

Analogously to this result, we have

**Proposition 2.3.** Let $X$ be (a) a CW-complex (resp. Whitehead complex), or (b) a paracompact space having the weak topology with respect to a point-countable closed covering of metric spaces. Then the following are equivalent.

1. $X$ is metrizable.
2. $X$ contains no copy of $S_2$ and no $S_\omega$ (resp. no copy of $S_2$).
3. $t(X \times S_c) \not\leq \omega$.

*Proof.* (1) $\Rightarrow$ (2) is easy. We have (3) $\Rightarrow$ (2) from Proposition 1.1.(2). (1) $\Rightarrow$ (3) follows from [2; Corollary 4].

(2) $\Rightarrow$ (1). In case of (b), we have this implication from Lemmas 1.2 and 2.1.

So, we suppose $X$ is a CW-complex. First we prove that $X$ is Fréchet. To see this, since $t(X) \not\leq \omega$, it is sufficient to show that every closed separable subset $S = \overline{D}$ with $D$ countable, is Fréchet. Clearly, $D$ is contained in some countable union $L$ of finite subcomplexes $L_n$. Since $L$ is closed in $X$, $S$ is a closed subset of $L$. Thus $S$ has the weak topology with respect to a countable covering of compact metric subsets $L_n \cap S$ of $S$. Since $S$ contains no copy of $S_2$, by Lemma 2.1(1), $S$ is Fréchet. Then $X$ is Fréchet. Second we prove that $X$ is metrizable. Since $X$ contains no copy of $S_\omega$, by Lemma 2.2, $X$ has the cells $\{e_\alpha\}$ such that
\{\overline{e}_\lambda\}, \ \overline{e}_\lambda = \text{cl} \ e_\lambda, \ \text{is point finite. For } x \in X, \ \text{let}

\{\overline{e}_\lambda; \overline{e}_\lambda \ni x\} = \{\overline{e}_{\lambda_1}, \overline{e}_{\lambda_2}, \ldots, \overline{e}_{\lambda_\ell}\}. \ \text{Put } E = \bigcup_{i=1}^\ell \overline{e}_{\lambda_i}.

Suppose x \in X - E. \ Since X is Fréchet, there is a convergent sequence \{x_n; n \in \mathbb{N}\} such that x_n \to x \text{ and } x_n \notin E. \ Since the convergent sequence is contained in a finite union of cells \overline{e}_\lambda, \ some \overline{e}_{\lambda_{i_0}} \text{ must contain an infinitely many } x_n \text{'s. Hence } x \in \overline{e}_{\lambda_{i_0}}. \ \text{Thus } \overline{e}_{\lambda_{i_0}} = \overline{e}_{\lambda_I}, \ \text{for some } i_0 \leq \ell. \ \text{But this is a contradiction, because } x_n \notin \overline{e}_{\lambda_{i_0}} \text{ for all } n. \ \text{Then } x \notin X - E, \ \text{which implies } x \in \text{int } E. \ Since E \text{ is compact metric, } X \text{ is locally metrizable. Then } X \text{ is metrizable, for } X \text{ is paracompact. Since a point-finite Whitehead complex is locally finite, the parenthetic part is proved similarly.}

Let I_\alpha be the space obtained from disjoint union of \alpha closed unit intervals [0,1] by identifying all zero points. Then each I_\alpha is a Whitehead complex. C. H. Dowker [4] showed that I_\omega \times I_\mathcal{C} \text{ is not a Whitehead complex.}

From Proposition 2.3 and Lemma 2.2, we have the following generalization of the Dowker's example.

**Corollary 2.4.** Let X \times Y be a CW-complex and \{e_\lambda; \lambda\} be the cells of Y. Then X is metrizable, or each cardinality of \{\lambda; \overline{e}_\lambda \ni y\} is less than c.

**Proposition 2.5.** Suppose that X_1 \text{ and } X_2 \text{ are CW-complexes (resp. Whitehead complexes). Then the following are equivalent.}

1. t(X_1 \times X_2) \leq \omega.
2. X_1 \times X_2 \text{ is a k-space.}
(3) $X_1 \times X_2$ is a CW-complex (resp. Whitehead complex).

Proof. (1) $\Rightarrow$ (2). Since $t(X_1 \times X_2) \leq \omega$, $X_1 \times X_2$ has the weak topology with respect to the closed covering of all closed, separable subsets of $X_1 \times X_2$. Each subset $S$ of $X_1 \times X_2$ is clearly contained in $\Pi_1(S) \times \Pi_2(S)$, where $\Pi_i : X_1 \times X_2 \rightarrow X_i (i = 1, 2)$ are projections. Thus $X_1 \times X_2$ has the weak topology with respect to a covering $\{F_1 \times F_2; F_i$ is a closed separable subset of $X_i \}$. As is seen in the proof of Proposition 2.3, (2) $\Rightarrow$ (1), each $F_i$ is a $k_\omega$-space. Hence, by [11; (7.5)] each $F_1 \times F_2$ is a k-space. This implies $X_1 \times X_2$ is a k-space.

(2) $\Rightarrow$ (3). Let $\{e_\gamma \} ; \{e_\delta \}$ be the cells of $X_1 ; X_2$ respectively. Since $X_1$ and $X_2$ are complexes; affine complexes, $X_1 \times X_2$ is a complex; affine complex with cells $\{e_\gamma \times e_\delta \}$ respectively. Moreover, if $X_1$ and $X_2$ are CW-complexes, then $X_1 \times X_2$ is closure finite. Thus, to prove that $X_1 \times X_2$ is a CW-complex (also, a Whitehead complex), we only show that $X_1 \times X_2$ has the weak topology with respect to a collection $\{\overline{e_\gamma} \times \overline{e_\delta} \}$. Each compact subset of $X_1 \times X_2$ is contained in a compact subset of $X_1 \times X_2$ with type $A \times B$. Then, each compact subset of $X_1 \times X_2$ is contained in a finite union of $\overline{e_\gamma} \times \overline{e_\delta}$. Since $X_1 \times X_2$ is a k-space, this implies that $X_1 \times X_2$ has the weak topology with respect to the collection $\{\overline{e_\gamma} \times \overline{e_\delta} \}$.

We have (3) $\Rightarrow$ (1) from that every CW-complex is sequential, hence $t(X_1 \times X_2) \leq \omega$.

Let $X$ be a CW-complex with the cells $\{e_\gamma \}$. Then we shall call $X$ point-finite; point-countable; locally
countable, if the covering \{e_{\gamma}\} of X is so respectively.

Lemma 2.6. Let X be a Fréchet CW-complex or a Whitehead complex. If X is a point-countable, then it is locally countable.

Proof. Since every point-countable Whitehead complex is locally countable, then we suppose that X is a Fréchet CW-complex. Let \{e_{\gamma}\} be the cells of X such that \{e_{\gamma}\} is point-countable. For \(x \in X\), let \(\{e_{\gamma}; e_{\gamma} \ni x\} = \{e_{\gamma_1}, e_{\gamma_2}, \ldots\}\). Put \(E = \bigcup_{i=1}^{\infty} e_{\gamma_i}\). Since X is Fréchet, by the proof of Proposition 2.3, (2) \(\Rightarrow\) (1), we have \(x \notin X - E\). This implies \(x \in \text{int } E\). Since each \(e_{\gamma_i}\) is compact, by the proof of [17; (D)], each \(e_{\gamma_i}\) meets at most finitely many \(e_{\gamma}\)'s, so that \(\text{int } E\) meets at most countably many \(e_{\gamma}\)'s. This implies that X is locally countable. The parenthetic part is proved similarly.

Proposition 2.7. Let X be a Fréchet CW-complex (resp. a Whitehead complex). Then the following are equivalent.

1. X is point-countable.
2. X is locally countable.
3. \(X^2\) is a CW-complex (resp. Whitehead complex).

Proof. (1) \(\Rightarrow\) (2) follows from Lemma 2.6.

(2) \(\Rightarrow\) (3). Every locally countable CW-complex is a \(k_\omega\)-space, and every product of two locally \(k_\omega\)-spaces is a \(k\)-space. Thus (2) \(\Rightarrow\) (3) follows from Proposition 2.5.

(3) \(\Rightarrow\) (1). Suppose that X is not point-countable. Then, by Lemma 2.2, X contains a closed copy of \(S_{\omega_1}\).
Thus $X^2$ is a $k$-space which contains a closed copy of $S_{\omega_1}^2$.

Hence $S_{\omega_1}^2$ is a $k$-space. However, by [7; Lemma 5], $S_{\omega_1}^2$ is not a $k$-space. This is a contradiction.

In terms of a set-theoretic axiom $BF(\omega_2)$ below, we shall consider the product $X \times Y$ of CW-complexes $X$ and $Y$.

$BF(\omega_2)$: If $F \subseteq \{f; f: N \rightarrow N \text{ is a function}\}$ has cardinality less than $\omega_2$, then there is a function $g: N \rightarrow N$ such that $\{n \in N; f(n) > g(n)\}$ is finite for all $f \in F$.

Hence CH implies $BF(\omega_2)$ is false.

In [7], Gary Gruenhage proved the following result (*):

(*) $S_{\omega_1} \times S_{\omega_1}$ is a $k$-space if and only if $BF(\omega_2)$ holds.

From this result (*), if the assertion of Proposition 1.1 by replacing "$S_{\omega_1}$" by "$S_{\omega_1}^c$" holds, then $BF(\omega_2)$ is false.

Lemma 2.8. $I_{\omega_1} \times I_{\omega_1}$ is a Whitehead complex if and only if $BF(\omega_2)$ holds.

Proof. "If." Since $BF(\omega_2)$ holds, by the proof of [7; Lemma 1] it turns out that $I_{\omega_1} \times I_{\omega_1}$ is sequential.

Hence $I_{\omega_1} \times I_{\omega_1}$ is a Whitehead complex by Proposition 2.5.

"Only if." $I_{\omega_1} \times I_{\omega_1}$ is a $k$-space and it contains a closed copy of $S_{\omega_1} \times S_{\omega_1}$, so that $S_{\omega_1} \times S_{\omega_1}$ is a $k$-space. Thus by the result (*), $BF(\omega_2)$ holds.

Proposition 2.9. If $X$ and $Y$ are Fréchet CW-complexes (resp. Whitehead complexes), then the following are equivalent.

(1) $X \times Y$ is a CW-complex (resp. Whitehead complex)
if and only if \(X\) or \(Y\) is locally finite, otherwise \(X\) and \(Y\) are locally countable.

(2) BF(\(\omega_2\)) is false.

Proof. (1) \(\Rightarrow\) (2) follows from Lemma 2.8.

(2) \(\Rightarrow\) (1). The "if" part of (1) does not use (2).

Suppose that \(X\) or \(Y\) is a locally finite CW-complex. Then \(X\) or \(Y\) is locally compact. Thus \(X \times Y\) is a k-space. Suppose that \(X\) and \(Y\) are locally countable. Then they are locally \(k_\omega\)-spaces, hence \(X \times Y\) is a k-space. In any case, \(X \times Y\) is a k-space. Hence \(X \times Y\) is a CW-complex by Proposition 2.5. The parenthetic part is proved similarly.

Next we prove the "only if" part. Suppose that \(Y\) is not locally countable. Then by Lemma 2.6, \(Y\) is not a point-countable CW-complex. Then by Lemma 2.2, \(Y\) contains a closed copy of \(S_{\omega_1}\). To show \(X\) is point-finite, suppose not. Then \(X\) contains a closed copy of \(S_\omega\) by Lemma 2.2. Thus \(X \times Y\) contains a closed copy of \(S_\omega \times S_{\omega_1}\). Since BF(\(\omega_2\)) is false, \(S_{\omega} \times S_{\omega_1}\) is not a k-space by the result (*). But, since \(X \times Y\) is a CW-complex, \(S_\omega \times S_{\omega_1}\) is a k-space. This is a contradiction. Thus \(X\) is point-finite, hence is locally finite by Lemma 2.6. Similarly, \(Y\) is locally finite if \(X\) is not locally countable. This finishes the proof.

The following questions (a) and (b) remain, the latter is related to Proposition 2.7.

Questions. (a) For every CW-complexes \(X\) and \(Y\), does \(1 \Rightarrow 2\) of the previous proposition hold?

(b) Is \(X\) locally countable if \(X^2\) is a CW-complex?
Supplement

Quite recently, through Zhou Hao-xuan, the author learned of the following result due to Liu Ying-ming: 'A necessary and sufficient condition for the product of CW-complexes,' Acta Mathematica Sinica, 21 (1978), 171-175 (Chinese).

[CH] Let X and Y be CW-complexes. Then $X \times Y$ is a CW-complex if and only if either X or Y is locally finite, or X and Y are locally countable.

Referring to the above paper and G. Gruenhage [7], we can prove that the answers to the questions (a) and (b) are affirmative.

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References


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