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1. Introduction

All spaces mentioned in this article are metrizable. Suppose X is an ANR, let f: X → Y be a (proper onto) cell-like map, and consider the following four statements.

(1) Y is an ANR.

(2) Y is countable dimensional.

(3) f is approximately invertible.

(4) f is a hereditary shape equivalence.

A by-now-classical theorem of Kozlowski [K] says that (1) and (4) are equivalent. The fact that (2) implies (4) was established for compact X in [K] and for general X in [A1]. The equivalence of (3) and (4) was verified for compact X in unpublished work of Kozlowski, and has recently been extended to a large class of non-locally compact X by [A2]. This article explores the extent to which these implications are valid if we assume that X is an approximate ANR.

To state our theorems efficiently, we introduce the following terminology. For functions f,g: X → Y and a collection $\mathcal{L}$ of subsets of Y, we say that f *is within* $\mathcal{L}$ *of* g if $\left\{\{f(x),g(x)\}: x \in X\right\}$ refines $\mathcal{L}$. Let $\mathcal{C}$ be a class of spaces; we say that a space X is an *approximate element* of $\mathcal{C}$ or is *approximately* of class $\mathcal{C}$ if for every open cover $\mathcal{L}$ of X, there is a Y ∈ $\mathcal{C}$ and maps α: X → Y and β: Y → X such that β ∘ α is within $\mathcal{L}$ of $1|X$. We will use this notion in two different instances: approximate ANR's, and approximately countable dimensional spaces.*

When X is an approximate ANR, we say that an onto map $f: X \to Y$ is *approximately invertible* if for every open cover $\mathcal{L}$ of Y, there is a map $g: Y \to X$ such that $g \circ f$ is within $f^{-1}\mathcal{L} = \{f^{-1}(L): L \in \mathcal{L}\}$ of $1|X$. Observe that $g \circ f$ is within $f^{-1}\mathcal{L}$ of $1|X$ if and only if $f \circ g$ is within $\mathcal{L}$ of $1|Y$. (The implication in one direction relies on the fact that f is onto.)

Recall that a map $f: X \to Y$ is *cell-like* if it is proper and onto and $f^{-1}(y)$ is a cell-like space for each $y \in Y$.**

We now state our theorems.

Theorem 1. *Let* X *be an approximate* ANR, *let* $f: X \to Y$ *be a cell-like map, and consider the following four statements.*

(1) Y *is an approximate* ANR.

(2) Y *is approximately countable dimensional.*

(3) f *is approximately invertible.*

(4) f *is a hereditary shape equivalence.*

Statements (1), (2) *and* (3) *are equivalent and are implied by statement* (4).

Theorem 2. *There is a cell-like map between approximate* ANR's *which is not a hereditary shape equivalence.*

2. The Proof of Theorem 1

Proof that (1) *implies* (2). Suppose Y is an approximate ANR. Let $\mathcal{L}$ be an open cover of Y. Then $\mathcal{L}$ is

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*A space is *countable dimensional* if it is the union of countably many finite dimensional subspaces.

**A space Z is *cell-like* if Z is compact and if every map of Z into an ANR is homotopic to a constant map.

star-refined* by an open cover $\mathcal{M}$ of Y. By hypothesis, there is an ANR W and maps $\alpha: Y \to W$ and $\beta: W \to Y$ such that $\beta \circ \alpha$ is within $\mathcal{M}$ of $1|Y$.

$\beta^{-1}\mathcal{M}$ is an open cover W. One of the fundamental properties of ANR's (Theorem 6.1 of [H]) provides a simplicial complex K and maps $\gamma: W \to |K|$ and $\delta: |K| \to W$ such that $\delta \circ \gamma$ is within $\beta^{-1}\mathcal{M}$ of $1|W$, where $|K|$ denotes the polyhedron underlying K. In the theorem just cited it is intended that $|K|$ be endowed with the Whitehead topology (p. 99 of [H]). However, since the Whitehead topology on $|K|$ may not be metrizable, and since we wish to work within the category of metrizable spaces, we endow $|K|$ with the metric topology (p. 100 of [H]) instead. The theorem cited above remains valid if $|K|$ is assigned the metric topology. The outline of the proof is unchanged; however certain details require additional care to insure the continuity of $\delta: |K| \to W$.

Since $|K| = \cup_{n=1}^{\infty}|K^n|$ where K^n is the n-skeleton of K, and $\dim|K^n| = n$ for each $n \geq 0$, then $|K|$ is countable dimensional. The maps $\gamma \circ \alpha: Y \to |K|$ and $\beta \circ \delta: |K| \to Y$ have the property that their composition $(\beta \circ \delta) \circ (\gamma \circ \alpha)$ is within $\mathcal{M}$ of $\beta \circ \alpha$. Hence $(\beta \circ \delta) \circ (\gamma \circ \alpha)$ is within $\mathcal{L}$ of $1|Y$. This shows that Y is approximately countable dimensional.

Proof that (2) *implies* (3). Since our proof relies on the Main Theorem of [A1], we must explain some of the

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*$\mathcal{M}$ *star-refines* $\mathcal{L}$ if for every $M \in \mathcal{M}$ there is an $L \in \mathcal{L}$ such that $\cup\{M' \in \mathcal{M}: M \cap M' \neq \emptyset\} \subset L$.

terminology occurring in [A1]. If $R \subset X \times Y$, we call R a *relation* from X to Y and we write $R: X \to Y$. If $R: X \to Y$ is a relation, then the *inverse* of R, denoted $R^{-1}: Y \to X$, is defined by $R^{-1} = \{(y,x) \in Y \times X: (x,y) \in R\}$. If $R: X \to Y$ and $S: Y \to Z$ are relations, then the *composition* of R and S, denoted $S \circ R: X \to Z$, is defined by $S \circ R = \{(x,z) \in X \times Z: (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in Y\}$. Suppose $R: X \to Y$ is a relation; for each $x \in X$, define $R(x) = \{y \in Y: (x,y) \in R\}$, and for each $A \subset X$, define $R(A) = \cup\{R(x): x \in A\}$. Thus, if $R: X \to Y$ is a relation, then $R^{-1}(y) = \{x \in X: (x,y) \in R\}$ for each $y \in Y$, and $R^{-1}(B) = \cup\{R^{-1}(y): y \in B\}$ for each $B \subset Y$. A relation $R: X \to Y$ is *continuous* if for every closed subset C of Y, $R^{-1}(C)$ is a closed subset of X. A relation $R: X \to Y$ is *cell-like* if it is continuous and if $R(x)$ is cell-like for each $x \in R^{-1}(Y)$.

One of the fundamental concepts in [A1] is that of a *slice-trivial* relation. For our purposes it is not necessary to state the full definition of slice-triviality. Instead, it suffices to know that each slice-trivial relation can be arbitrarily closely approximated by maps. More precisely:

Proposition 3. *Every slice-trivial relation* $R: X \to Y$ *has the following property. For every collection* $\mathcal{L}$ *of open subsets of* Y *which is refined by* $\{R(x): x \in X\}$, *there is a map* $f: R^{-1}(Y) \to Y$ *which is within* $\mathcal{L}$ *of* R; i.e., $\{R(x) \cup \{f(x)\}: x \in R^{-1}(Y)\}$ *refines* $\mathcal{L}$.

We now state the special case of the Main Theorem of [A1] which we shall need here.

Theorem 4. *If* R: X → Y *is a cell-like relation from a countable dimensional space* X *to an* ANR Y, *then* R *is slice-trivial.*

We also need the following.

Lemma 5. *Every approximate* ANR X *has the following property. If* i: X → W *is a closed embedding of* X *into a metric space* W, *then for every open cover* $\mathcal{L}$ *of* X, *there is an open neighborhood* O *of* i(X) *in* W *and a map* ψ: O → X *such that* $\psi \circ i$ *is within* $\mathcal{L}$ *of* $1|X$.

Proof. Let $\mathcal{L}$ be an open cover of X. Then there is an ANR Z and maps α: X → Z and β: Z → X such that $\beta \circ \alpha$ is within $\mathcal{L}$ of $1|X$. If i: X → W is a closed embedding into a metric space W, then there is an open neighborhood O of i(X) in W and a map γ: O → Z such that $\gamma \circ i = \alpha$. Define the map ψ: O → X by $\psi = \beta \circ \gamma$. Then $\gamma \circ i = \beta \circ \alpha$.

We now prove that (2) implies (3). Assume Y is approximately countable dimensional. Let $\mathcal{L}$ be an open cover of Y. We shall produce a map g: Y → X such that $g \circ f$ is within $f^{-1}\mathcal{L}$ of $1|X$.

There are open covers $\mathcal{M}$ and $\mathcal{N}$ of Y such that $\mathcal{M}$ star-refines $\mathcal{L}$ and $\mathcal{N}$ star-refines $\mathcal{M}$. There is a countable dimensional space Z and maps α: Y → Z and β: Z → Y such that $\beta \circ \alpha$ is within $\mathcal{M}$ of $1|Y$. Let i: X → W be a closed embedding of X in an ANR W. Then Lemma 5 provides an open neighborhood O of i(X) in W and a map ψ: O → X such that $\psi \circ i$ is within $f^{-1}\mathcal{N}$ of $1|X$. It follows that for each $y \in Y$, $if^{-1}(y) \subset (f \circ \psi)^{-1}(\cup\{N \in \mathcal{N}: y \in N\})$. Therefore, $\{if^{-1}(y): y \in Y\}$

refines $(f \circ \psi)^{-1}(\mathcal{M})$.

$$\begin{array}{ccc} O & \overset{\phi}{\longleftarrow - - -} & Z \\ \psi \downarrow \uparrow i & & \alpha \uparrow \downarrow \beta \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Theorem 4 implies that the cell-like relation $i \circ f^{-1} \circ \beta: Z \to O$ is slice-trivial. Since $\{i \circ f^{-1} \circ \beta(z): z \in Z\}$ refines $(f \circ \psi)^{-1}(\mathcal{M})$, then Proposition 3 provides a map $\phi: Z \to O$ which is within $(f \circ \psi)^{-1}(\mathcal{M})$ of $i \circ f^{-1} \circ \beta$. Define the map $g: Y \to X$ by $g = \psi \circ \phi \circ \alpha$.

It remains to verify that $g \circ f$ is within $f^{-1}\mathcal{L}$ of $1|X$. Let $x \in X$. There is an $M' \in \mathcal{M}$ such that $(f \circ \psi)^{-1}(M')$ contains $\phi(\alpha \circ f(x))$ and $i \circ f^{-1} \circ \beta(\alpha \circ f(x))$. Hence, $f^{-1}(M')$ contains $g \circ f(x)$ and $\psi \circ i \circ f^{-1} \circ \beta \circ \alpha \circ f(x)$. Let $x' \in f^{-1} \circ \beta \circ \alpha \circ f(x)$. Then $\psi \circ i(x') \in f^{-1}(M')$. There is an $M \in \mathcal{M}$ such that $f^{-1}(M)$ contains x' and $\psi \circ i(x')$. Then $\beta \circ \alpha \circ f(x) = f(x') \in M$ and $f \circ \psi \circ i(x') \in M \cap M'$. Finally there is an M'' which contains both $f(x)$ and $\beta \circ \alpha \circ f(x)$. Thus, $x \in f^{-1}(M'')$ and $\beta \circ \alpha \circ f(x) \in M \cap M''$. Since $M \cap M' \neq 0$ and $M \cap M'' \neq 0$, then $M \cup M' \cup M'' \subset L$ for some $L \in \mathcal{L}$. Since $g \circ f(x) \in f^{-1}(M')$ and $x \in f^{-1}(M'')$, then $\{x, g \circ f(x)\} \subset f^{-1}(L)$.

Proof that (3) *implies* (1). Assume that $f: X \to Y$ is approximately invertible. Let $\mathcal{L}$ be an open cover of Y. Then there is an open cover $\mathcal{M}$ of Y which star-refines $\mathcal{L}$. By hypothesis, there is an ANR W and maps $\alpha: X \to W$ and $\beta: W \to X$ such that $\beta \circ \alpha$ is within $f^{-1}\mathcal{M}$ of $1|X$. Also there is a map $g: Y \to X$ such that $g \circ f$ is within $f^{-1}\mathcal{M}$ of $1|X$. It is easy to verify that the maps $\alpha \circ g: Y \to W$ and

$f \circ \beta$: $W \to Y$ have the property that their composition $(f \circ \beta) \circ (\alpha \circ g)$ is within $\mathcal{M}$ of $f \circ g$. It is also easy to see that $f \circ g$ is within $\mathcal{M}$ of $1|Y$. Hence, $(f \circ \beta) \circ (\alpha \circ g)$ is within $\mathcal{L}$ of $1|Y$. This proves that Y is an approximate ANR.

Proof that (4) *implies* (3). The original definition of "hereditary shape equivalence" is presented in [K] in a form which can't be used directly here. So rather than stating it, we shall describe one of its more useful implications. Lemmas 5 and 6 of [K] entail the following.

Proposition 6. *If a proper onto map* f: $X \to Y$ *is a hereditary shape equivalence, then it has the following property. If* α: $X \to W$ *is a map of* X *into an* ANR W, *and if* $\mathcal{O}$ *is a collection of open subsets of* W *which is refined by* $\{\alpha(f^{-1}(y))$: $y \in Y\}$, *then there is a map* γ: $Y \to W$ *such that* $\gamma \circ f$ *is within* $\mathcal{O}$ *of* a.

Now assume that f: $X \to Y$ is a hereditary shape equivalence. Let L be an open cover of Y. Select open covers M and N of Y such that M star-refines L and N star-refines M. By hypothesis there is an ANR W and maps α: $X \to W$ and β: $W \to X$ such that $\beta \circ \alpha$ is within $f^{-1}N$ of $1|X$. It follows that for each $y \in Y$, $\alpha(f^{-1}(y)) \subset (f \circ \beta)^{-1}(\cup\{N \in N$: $y \in N\})$. Therefore, $\{\alpha(f^{-1}(y))$: $y \in Y\}$ refines $(f \circ \beta)^{-1}(M)$. Proposition 6 now provides a map γ: $Y \to W$ such that $\gamma \circ f$ is within $(f \circ \beta)^{-1}(M)$ of α. Define the map g: $Y \to X$ by $g = \beta \circ \gamma$. It follows easily that $g \circ f$ is within $f^{-1}M$ of $\beta \circ \alpha$. Since $\beta \circ \alpha$ is within $f^{-1}M$ of $1|X$, we conclude that $g \circ f$ is within $f^{-1}L$ of $1|X$. This proves that $f: X \to Y$ is approximately invertible.

3. The Proof of Theorem 2

We shall construct a cell-like map $f: X \to Y$ which is not a hereditary shape equivalence, but where both X and Y are approximate ANR's. J. Segal has called to the authors' attention the similarity between this example and the construction on page 223 of [KS]. Also see [DK]. At the heart of our example is Taylor's remarkable cell-like map $\tau: T \to \mathbf{Q}$ which is not a shape equivalence, where $\mathbf{Q}$ is the Hilbert cube and T is a compact metric space which is not cell-like [T]. Results from [A2] show that T is not an approximate ANR. (See the remark following this proof.)

We begin by embedding T in an approximate ANR which is in some sense a minimal enlargement of T. We assert that there is a compact approximate ANR X which is the disjoint union of T and a countable collection of compact polyhedra $\{P_i\}$, such that for each neighborhood U of T in X, there is an $n \geq 1$ such that $\cup_{i=n+1}^{\infty} P_i \subset U$. (The construction of X described below can be carried out with any compact metric space in place of T.)

According to [F], T is homeomorphic to the inverse limit of an inverse sequence $\{P_i, f_{i,j}\}$ where each P_i is a compact polyhedron. Hence, there is a homeomorphism e_∞ from T onto the subset

$$\{(p_i) \in \Pi_{i=1}^{\infty} P_i: f_{i,j}(p_i) = p_j \text{ for } i \leq j \leq i\} \times \{0\}$$

of $(\Pi_{i=1}^{\infty} P_i) \times [0,1]$. We construct X in $(\Pi_{i=1}^{\infty} P_i) \times [0,1]$. Fix a point (q_i) of $\Pi_{i=1}^{\infty} P_i$. For each $n \geq 1$, define the embedding $e_n: P_n \to (\Pi_{i=1}^{\infty}P_i) \times [0,1]$ by $e_n(p) = (f_{n,1}(p),\cdots, f_{n,n-1}(p),p,q_{n+1},q_{n+2},\cdots) \times (1/n)$ for $p \in P_n$. Let $X = e_\infty(T) \cup (\cup_{i=1}^{\infty}e_i(P_i))$. It is easy to verify that if U is a neighborhood of $e_\infty(T)$ in X, then $\cup_{i=n+1}^{\infty}e_i(P_i) \subset U$ for some $n \geq 1$. To show that X is an appropriate ANR, we define for each $n \geq 1$ a map $r_n: (\Pi_{i=1}^{\infty}P_i) \times [0,1] \to (\Pi_{i=1}^{\infty}P_i) \times [0,1]$ by $r_n((p_i) \times t) = (p_1,\cdots,p_n,q_{n+1},q_{n+2},\cdots) \times \max\{t,1/n\}$ for $(p_i) \times t \in (\Pi_{i=1}^{\infty}P_i) \times [0,1]$. Then for each $n \geq 1$, $r_n|X$ is a retraction of X onto the ANR $\cup_{i=1}^{n}e_i(P_i)$; and $\{r_n\}$ converges uniformly to $1|(\Pi_{i=1}^{\infty}P_i) \times [0,1]$. Hence, if $\mathcal{L}$ is an open cover of X, there is an $n \geq 1$ such that the composition of $r_n|X: X \to \cup_{i=1}^{n}e_i(P_i)$ and the inclusion of $\cup_{i=1}^{n}e_i(P_i)$ in X is within $\mathcal{L}$ of $1|X$. Finally, we identify T with $e(T)$ and P_i with $e_i(P_i)$ for $i \geq 1$, to make X the disjoint union of T and the P_i's.

Let Y be the space obtained by attaching X to $\mathbf{Q}$ via the map $\tau: T \to \mathbf{Q}$; i.e., $Y = X \cup_\tau \mathbf{Q}$. Then $\tau: T \to \mathbf{Q}$ extends naturally to a cell-like map $f: X \to Y$ such that $f|\cup_{i=1}^{\infty}P_i$ is a homeomorphism of $X - T$ onto $Y - \mathbf{Q}$.

Y is a compact metric space which is the disjoint union of $\mathbf{Q}$ and the countable collection of compact polyhedra $\{f(P_i)\}$. Furthermore, for each neighborhood U of $\mathbf{Q}$ in Y, there is an $n \geq 1$ such that $\cup_{i=n+1}^{\infty}f(P_i) \subset U$. To see that Y is an approximate ANR, let $\mathcal{L}$ be an open cover of Y. Since $\mathbf{Q}$ is an absolute retract, there is a retraction $r: Y \to \mathbf{Q}$. $\mathbf{Q}$ has a neighborhood U in Y such that $\left\{\{y,r(y)\}: y \in U\right\}$ refines $\mathcal{L}$. Choose $n \geq 1$ so that $\cup_{i=n+1}^{\infty}f(P_i) \subset U$. Then a retraction ρ of Y onto the ANR $\mathbf{Q} \cup (\cup_{i=1}^{n}f(P_i))$ is defined by

$$\rho(y) = \begin{cases} r(y) & \text{if } y \in \mathfrak{Q} \cup (\cup_{i=n+1}^{\infty} f(P_i)) \\ y & \text{if } y \in \cup_{i=1}^{n} f(P_i) \end{cases}$$

for $y \in Y$. Furthermore, the composition of ρ and the inclusion of $\mathfrak{Q} \cup (\cup_{i=1}^{n} f(P_i))$ into Y is within L of $1|Y$.

The cell-like map $f: X \to Y$ is not a hereditary shape equivalence because $f|T = \tau$ is not a shape equivalence. Indeed, according to the definition of hereditary shape equivalence in [K], $f: X \to Y$ is a hereditary shape equivalence if and only if $f|f^{-1}(C): f^{-1}(C) \to C$ is a shape equivalence for each closed subset C of Y.

One might wonder whether an example of this type can be constructed in which one of X and Y is an ANR and the other is an approximate ANR. Results of [A2] rule out this possibility: if $f: X \to Y$ is a cell-like map where one of X and Y is an ANR and the other is an approximate ANR, then f is a hereditary shape equivalence.

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